अध्याय-7

समाकलन

(Integrals)

(Important Formulae and Definitions)

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + c, (n \neq -1)$

2. $\int \frac{1}{x} dx = \log_e |x| + c$

3. $\int e^x dx = e^x + c$

4. $\int a^x dx = \frac{a^x}{\log_e a} + c$

5. $\int \sin x\, dx = -\cos x + c$

6. $\int \cos x\, dx = \sin x + c$

7. $\int \sec^2 x\, dx = \tan x + c$

8. $\int \operatorname{cosec}^2 x\, dx = -\cot x + c$

9. $\int \sec x \tan x\, dx = \sec x + c$

10. $\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + c$

11. $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$

12. $\int \frac{1}{\sqrt{1+x^2}} dx = \tan^{-1} x + c$

13. $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + c$

14. $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$

15. $\int \frac{dx}{ax+b} = \frac{1}{a} \log|ax+b| + c$

16. $\int e^{ax} dx = \frac{1}{a} e^{ax} + c$

17. $\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$

18. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a} + c$

19. $\int \frac{dx}{x\sqrt{x^2-a^2}} dx = \frac{1}{a} \sec^{-1} \frac{x}{a} + c$

20. $\int \tan x\, dx = -\log|\cos x| + c$ या $\log|\sec x| + c$

21. $\int \cot x\, dx = \log|\sin x| + c$

22. $\int \text{cosec}\, x\, dx = \log\left|\tan \frac{x}{2}\right| + c = \log|\text{cosec}\, x - \cot x| + c$

23. $\int \sec x\, dx = \log|\sec x + \tan x| + c = \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + c$

24. $\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx) + c$

25. $\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2 + b^2}(b \sin bx + a \cos bx) + c$

26. $\int \sqrt{a^2 - x^2}\, dx = \frac{1}{2}\left[x\sqrt{a^2 - x^2} + a^2 \sin^{-1} \frac{x}{a}\right] + c$

27. $\int \sqrt{a^2 + x^2}\, dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2} \log_e |x + \sqrt{a^2 + x^2}| + c$

28. $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log_e |x + \sqrt{x^2 - a^2}| + c$

29. $\int \frac{1}{x^2 - a^2}\, dx = \frac{1}{2a} \log_e \left|\frac{x - a}{x + a}\right| + c$, जब $x > a$

30. $\int \frac{1}{a^2 - x^2}\, dx = \frac{1}{2a} \log_e \left|\frac{x + a}{x - a}\right| + c$, जब $x < a$

31. $\int \frac{dx}{ax^2 + bx + c} = \frac{2}{\sqrt{4ac - b^2}} \tan^{-1} \frac{2ax + b}{\sqrt{4ac - b^2}}$, जब $4ac > b^2$

$$= -\frac{1}{\sqrt{b^2 - 4ac}} \log_e \frac{2ax + b - \sqrt{b^2 - 4ac}}{2ax + b + \sqrt{b^2 - 4ac}}, \text{ जब } 4ac < b^2.$$

प्रश्नावली 7·1

निम्नलिखित फलनों के प्रतिअवकलज (समाकलन) निरीक्षण विधि द्वारा ज्ञात कीजिए :

प्रश्न 1. sin 2x.

हल : हम जानते हैं कि

$$\frac{d}{dx} \cos 2x = -2 \sin 2x$$

$$\therefore \quad -\frac{1}{2} \frac{d}{dx} \cos 2x = \sin 2x$$

$$\therefore \quad \int \sin 2x\, dx = -\frac{1}{2} \cos 2x + C.$$ उत्तर

प्रश्न 2. cos 3x.

हल : हम जानते हैं कि

$$\frac{d}{dx}\sin 3x = 3\cos 3x$$

$$\therefore \qquad \cos 3x = \frac{d}{dx}\left(\frac{1}{3}\sin 3x\right)$$

$$\therefore \qquad \int \cos 3x\, dx = \frac{1}{3}\sin 3x + C.$$ उत्तर

प्रश्न 3. e^{2x}.

हल : हम जानते हैं कि

$$\frac{d}{dx}e^{2x} = 2e^{2x}$$

$$e^{2x} = \frac{d}{dx}\left(\frac{1}{2}e^{2x}\right)$$

$$\therefore \qquad \int e^{2x}\, dx = \frac{1}{2}e^{2x} + C.$$ उत्तर

प्रश्न 4. $(ax + b)^2$.

हल : हम जानते हैं कि

$$\frac{d}{dx}(ax+b)^3 = 3a(ax+b)^2$$

या $$(ax+b)^2 = \frac{d}{dx}\left(\frac{1}{3a}(ax+b)^3\right)$$

या $$\int (ax+b)^2\, dx = \frac{1}{3a}(ax+b)^3 + C.$$ उत्तर

प्रश्न 5. sin 2x – 4e^{3x}

हल : हम जानते हैं कि

$$\frac{d}{dx}\cos 2x = -2\sin 2x$$

या $$\sin 2x = \frac{d}{dx}\left(-\frac{1}{2}\cos 2x\right)$$

$$\therefore \qquad \int \sin 2x\, dx = -\frac{1}{2}\cos 2x + C_1 \qquad \text{...(i)}$$

और $$\frac{d}{dx}(e^{3x}) = 3e^{3x} \text{ या } \frac{d}{dx}\left(\frac{1}{3}e^{3x}\right) = e^{3x}$$

$$\therefore \qquad \int e^{3x}\, dx = \frac{1}{3}e^{3x} + C_2 \qquad \text{...(ii)}$$

समी. (i) तथा (ii) से,

$$\int (\sin 2x - 4e^{3x})\, dx = -\frac{1}{2}\cos 2x - \frac{4}{3}e^{3x} + C \qquad [\because C = C_1 + C_2]$$ उत्तर

निम्नलिखित समाकलनों को ज्ञात कीजिए—

प्रश्न 6. $\int (4e^{3x}+1)\,dx$.

हल :
$$\int (4e^{3x}+1)\,dx = 4\int e^{3x}\,dx + \int dx$$
$$= \frac{4}{3}e^{3x} + x + C.$$
उत्तर

प्रश्न 7. $\int x^2\left(1-\frac{1}{x^2}\right)dx$.

हल :
$$\int x^2\left(1-\frac{1}{x^2}\right)dx = \int\left(x^2 - x^2.\frac{1}{x^2}\right)dx = \int (x^2-1)\,dx$$
$$= \int x^2dx - \int dx$$
$$= \frac{x^3}{3} - x + C.$$
उत्तर

प्रश्न 8. $\int (ax^2+bx+c)\,dx$.

हल :
$$\int (ax^2+bx+c)\,dx = a\int x^2dx + b\int x\,dx + c\int dx$$
$$= a.\frac{x^3}{2} + b.\frac{x^2}{2} + cx + C$$
$$= \frac{ax^3}{3} + \frac{bx^2}{2} + cx + C.$$
उत्तर

प्रश्न 9. $\int (2x^2+e^x)\,dx$.

हल :
$$2\int x^2dx + \int e^x dx = 2\times\frac{x^3}{3} + e^x + c$$
$$= \frac{2}{3}x^3 + e^x + c.$$
उत्तर

प्रश्न 10. $\int\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2 dx$.

हल :
$$\int\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2 dx = \int\left[(\sqrt{x})^2 - 2\sqrt{x}.\frac{1}{\sqrt{x}} + \left(\frac{1}{\sqrt{x}}\right)^2\right]dx$$
$$= \int\left(x - 2 + \frac{1}{x}\right)dx = \int x\,dx - 2\int dx + \int\frac{1}{x}dx$$
$$= \frac{x^2}{2} - 2x + \log|x| + C.$$
उत्तर

प्रश्न 11. $\int\frac{x^3+5x^2-4}{x^2}dx$.

हल :
$$\int\frac{x^3+5x^2-4}{x^2}dx = \int\left(\frac{x^3}{x^2}+\frac{5x^2}{x^2}-\frac{4}{x^2}\right)dx = \int\left(x+5-\frac{4}{x^2}\right)dx$$
$$= \int x\,dx + 5\int dx - 4\int\frac{1}{x^2}dx$$

$$= \int x\,dx + 5\int dx - 4\int x^{-2}dx$$

$$= \frac{x^2}{2} + 5x - \frac{4x^{-2+1}}{-2+1} + C$$

$$= \frac{x^2}{2} + 5x - \frac{4x^{-1}}{-1} + C$$

$$= \frac{x^2}{2} + 5x + \frac{4}{x} + C.$$ उत्तर

प्रश्न 12. $\int \frac{x^3 + 3x + 4}{\sqrt{x}}\,dx$.

हल :

$$\int \frac{x^3 + 3x + 4}{\sqrt{x}}\,dx = \int \left(\frac{x^3}{\sqrt{x}} + \frac{3x}{\sqrt{x}} + \frac{4}{\sqrt{x}}\right) dx$$

$$= \int (x^{\frac{5}{2}} + 3x^{\frac{1}{2}} + 4x^{-\frac{1}{2}})\,dx$$

$$= \int x^{\frac{5}{2}}\,dx + 3\int x^{\frac{1}{2}}\,dx + 4\int x^{-\frac{1}{2}}\,dx$$

$$= \frac{x^{\frac{5}{2}+1}}{\frac{5}{2}+1} + \frac{3x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{4x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}$$

$$= \frac{2}{7}x^{\frac{7}{2}} + 3 \times \frac{2}{3}x^{\frac{3}{2}} + 4.\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= \frac{2}{7}x^{\frac{7}{2}} + 2x^{\frac{3}{2}} + 8\sqrt{x} + C.$$ उत्तर

प्रश्न 13. $\int \frac{x^3 - x^2 + x - 1}{x - 1}\,dx$.

हल :

$$\int \frac{x^3 - x^2 + x - 1}{x - 1}\,dx = \int \frac{x^2(x-1) + 1(x-1)}{x-1}\,dx$$

$$= \int \frac{(x-1)(x^2+1)}{x-1}\,dx = \int (x^2 + 1)\,dx$$

$$= \int x^2\,dx + \int dx$$

$$= \frac{x^3}{3} + x + C.$$ उत्तर

प्रश्न 14. $\int (1 - x)\sqrt{x}\,dx$.

हल :

$$\int (1 - x)\sqrt{x}\,dx = \int \left(x^{\frac{1}{2}} - x^{\frac{3}{2}}\right) dx = \int x^{\frac{1}{2}}\,dx - \int x^{\frac{3}{2}}\,dx$$

$$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + C$$

$$= \frac{2}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}} + C.$$ उत्तर

प्रश्न 15. $\int \sqrt{x}(3x^2 + 2x + 3)\,dx.$

हल : $$\int \sqrt{x}(3x^2 + 2x + 3)\,dx = \int (3x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + 3x^{\frac{1}{2}})\,dx$$

$$= 3\int x^{\frac{5}{2}}\,dx + 2\int x^{\frac{3}{2}}\,dx + 3\int x^{\frac{1}{2}}\,dx$$

$$= 3.\frac{x^{\frac{5}{2}+1}}{\frac{5}{2}+1} + 2\frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + 3.\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$$

$$= 3 \times \frac{2}{7}x^{\frac{7}{2}} + 2 \times \frac{2}{5}x^{\frac{5}{2}} + 3 \times \frac{2}{3}x^{\frac{3}{2}} + C$$

$$= \frac{6}{7}x^{\frac{7}{2}} + \frac{4}{5}x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + C.$$ उत्तर

प्रश्न 16. $\int (2x - 3\cos x + e^x)\,dx.$

हल : $$\int (2x - 3\cos x + e^x)\,dx = 2\int x\,dx - 3\int \cos x\,dx + \int e^x\,dx$$

$$= 2.\frac{x^2}{2} - 3\sin x + e^x + C$$

$$= x^2 - 3\sin x + e^x + C.$$ उत्तर

प्रश्न 17. $\int (2x^2 - 3\sin x + 5\sqrt{x})\,dx.$

हल : $$\int (2x^2 - 3\sin x + 5\sqrt{x})\,dx = 2\int x^2\,dx - 3\int \sin x\,dx + 5\int \sqrt{x}\,dx$$

$$= 2 \times \frac{x^3}{3} - 3(-\cos x) + \frac{5.x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$$

$$= \frac{2}{3}x^3 + 3\cos x + 5 \times \frac{2}{3}x^{\frac{3}{2}} + C$$

$$= \frac{2}{3}x^3 + 3\cos x + \frac{10}{3}x^{\frac{3}{2}} + C.$$ उत्तर

प्रश्न 18. $\int \sec x (\sec x + \tan x)\, dx.$

हल :

$$\int \sec x\, (\sec x + \tan x)\, dx = \int (\sec^2 x + \sec x \tan x)\, dx$$

$$= \int \sec^2 x\, dx + \int \sec x \tan x\, dx$$

$$= \tan x + \sec x + C.$$

उत्तर

प्रश्न 19. $\int \frac{\sec^2 x}{\operatorname{cosec}^2 x} dx.$

हल :

$$\int \frac{\sec^2 x}{\operatorname{cosec}^2 x} dx = \int \frac{\sin^2 x}{\cos^2 x} = \int \tan^2 x\, dx$$

$$= \int (\sec^2 x - 1)\, dx$$

$$= \int \sec^2 x\, dx - \int dx$$

$$= \tan x - x + C.$$

उत्तर

प्रश्न 20. $\int \frac{2 - 3\sin x}{\cos^2 x} dx.$

हल :

$$\int \frac{2 - 3\sin x}{\cos^2 x} dx = \int \frac{2}{\cos^2 x} dx - \int \frac{3\sin x}{\cos^2 x} dx$$

$$= 2\int \frac{1}{\cos^2 x} dx - 3\int \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} dx$$

$$= 2\int \sec^2 x\, dx - 3\int \sec x \tan x\, dx$$

$$= 2 \tan x - 3 \sec x + C.$$

उत्तर

प्रश्न 21 व 22 में सही उत्तर का चयन कीजिए —

प्रश्न 21. $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$ **का प्रतिअवकलज है :**

(A) $\frac{1}{3}x^{\frac{1}{3}} + 2x^{\frac{1}{2}} + C$

(B) $\frac{2}{3}x^{\frac{2}{3}} + \frac{1}{2}x^2 + C$

(C) $\frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} +$

(D) $\frac{3}{2}x^{\frac{3}{2}} + \frac{1}{2}x^{\frac{1}{2}} + C$

हल :

$$\int\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)dx = \int x^{1/2}dx + \int x^{-1/2}dx$$

$$= \frac{x^{3/2}}{\frac{3}{2}} + \frac{x^{1/2}}{\frac{1}{2}} + C$$

$$= \frac{2}{3}x^{3/2} + 2x^{1/2} + C$$

अत: विकल्प (C) सही है।

उत्तर

प्रश्न 22. यदि $\frac{d}{dx}f(x) = 4x^3 - \frac{3}{x^4}$ जिसमें $f(2) = 0$ तो $f(x)$ है :

(A) $x^4 + \frac{1}{x^3} - \frac{129}{8}$ (B) $x^3 + \frac{1}{x^4} + \frac{129}{8}$

(C) $x^4 + \frac{1}{x^3} + \frac{129}{8}$ (D) $x^3 + \frac{1}{x^4} - \frac{129}{8}$

हल : ∵ $\frac{d}{dx}f(x) = 4x^3 - \frac{3}{x^4}$

या $\left(4x^3 - \frac{3}{x^4}\right) = f(x)$ का प्रति अवकलज

∴ $f(x) = \int\left(4x^3 - \frac{3}{x^4}\right)dx$

$= 4\int x^3 dx - 3\int x^{-4} dx$

$= 4 \times \frac{x^4}{4} - 3 \times \frac{x^{-3}}{-3} + C$

$f(x) = x^4 + x^{-3} + C$

∵ $f(2) = 0$ (दिया है)

∴ $f(2) = (2)^4 + (2)^{-3} + C = 16 + \frac{1}{8} + C$

$\frac{129}{8} + C = 0$ या $C = -\frac{129}{8}$

∴ $f(x) = x^4 + x^{-3} - \frac{129}{8}$

$f(x) = x^4 + \frac{1}{x^3} - \frac{129}{8}$

अतः विकल्प (A) सही है। **उत्तर**

प्रश्नावली 7·2

1 से 37 तक के प्रत्येक फलन का समाकलन ज्ञात कीजिए—

प्रश्न 1. $\frac{2x}{1 + x^2}$.

हल : $\int \frac{2x}{1 + x^2} dx$

∵ $1 + x^2 = t$ रखने पर

∴ $2x\,dx = dt$

अत:
$$\int \frac{2x}{1+x^2}\,dx = \int \frac{dt}{t} = \log|t| + C$$
$$= \log(1+x^2) + C$$
(t का मान रखने पर) उत्तर

प्रश्न 2. $\frac{(\log x)^2}{x}$.

हल :
$$\int \frac{(\log x)^2}{x}\,dx$$
$\because$ $\log x = t$ रखने पर

$\therefore$
$$\frac{1}{x}\,dx = dt$$
अत:
$$\int \frac{(\log x)^2}{x}\,dx = \int t^2\,dt = \frac{t^3}{3} + C$$
$$= \frac{1}{3}(\log x)^3 + C.$$
(t का मान रखने पर) उत्तर

प्रश्न 3. $\frac{1}{x + x\log x}$.

हल :
$$\int \frac{1}{x + x\log x}\,dx = \int \frac{1}{x(1+\log x)}\,dx$$
$\because$ $1 + \log x = t$ प्रतिस्थापित करने पर

$\therefore$
$$\frac{1}{x}\,dx = dt$$
अत:
$$\int \frac{1}{x(1+\log x)}\,dx = \int \frac{1}{t}\,dt = \log|t| + C$$
$$= \log|1 + \log x| + C.$$
(t का मान रखने पर) उत्तर

प्रश्न 4. $\sin x \sin(\cos x)$.

हल :
$$\int \sin x \sin(\cos x)\,dx$$
$\because$ $\cos x = t$ प्रतिस्थापित करने पर

$\therefore$
$$-\sin x\,dx = dt$$
अत:
$$-\int \sin t\,dt = \cos t + C$$
$$= \cos(\cos x) + C.$$
(t का मान रखने पर) उत्तर

प्रश्न 5. $\sin(ax+b)\cos(ax+b)$.

हल :
$$\int \sin(ax+b)\cos(ax+b)\,dx$$
$\because$ $\sin(ax+b) = t$ रखने पर

$\therefore$
$$a\cos(ax+b)dx = dt$$
अत:
$$\int \sin(ax+b)\cos(ax+b)\,dx = \frac{1}{a}\int t\,dt = \frac{1}{a}\cdot\frac{t^2}{2} + C$$
$$= \frac{\sin^2(ax+b)}{2a} + C.$$
(t का मान रखने पर) उत्तर

प्रश्न 6. $\sqrt{ax+b}$.

हल : $$\int \sqrt{ax+b}\ dx$$

$\because$ $ax + b = t$ प्रतिस्थापित करने पर

$\therefore$ $$a\,dx = dt$$

अत: $$\int \sqrt{ax+b}\ dx = \frac{1}{a}\int \sqrt{t}\ dt = \frac{1}{a}\int t^{1/2}\ dt$$

$$= \frac{1}{a}\left(\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right) + C = \frac{2}{3a}t^{\frac{3}{2}} + C$$

$$= \frac{2}{3a}(ax+b)^{\frac{3}{2}} + C.$$ (t का मान रखने पर) **उत्तर**

प्रश्न 7. $x\sqrt{x+2}$.

हल : $$\int x\sqrt{x+2}\ dx$$

$$= \int (x+2-2).\sqrt{x+2}\ dx$$

$$= \int (x+2)^{\frac{3}{2}} dx - 2\int \sqrt{x+2}\ dx$$

$$= \frac{(x+2)^{\frac{3}{2}+1}}{\frac{3}{2}+1} - 2.\frac{(x+2)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$$

$$= \frac{2}{5}(x+2)^{\frac{5}{2}} - \frac{4}{3}(x+2)^{\frac{3}{2}} + C.$$ **उत्तर**

प्रश्न 8. $x\sqrt{1+2x^2}$.

हल : $$\int x\sqrt{1+2x^2}dx$$

$\because$ $1 + 2x^2 = t$ रखने पर

$$4x\,dx = dt$$

$\therefore$ $$\int x\sqrt{1+2x^2}\ dx = \frac{1}{4}\int \sqrt{t}\ dt$$

$$= \frac{1}{4}.\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} = \frac{1}{6}t^{\frac{3}{2}} + C$$

$$= \frac{1}{6}(1+2x^2)^{\frac{3}{2}} + C.$$ (t का मान रखने पर) **उत्तर**

प्रश्न 9. $(4x+2)\sqrt{x^2+x+1}$.

हल : $$\int(4x+2)\sqrt{x^2+x+1}\,dx = 2\int(2x+1)\sqrt{x^2+x+1}\,dx$$

$\because$ $x^2+x+1=t$ रखने पर

$\therefore$ $(2x+1)\,dx = dt$

$\therefore$ $$= 2\int t^{\frac{1}{2}}dt = 2.\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1}+C$$

$$= \frac{4}{3}t^{\frac{3}{2}}+C = \frac{4}{3}(x^2+x+1)^{\frac{3}{2}}+C.$$

(t का मान रखने पर) **उत्तर**

प्रश्न 10. $\dfrac{1}{x-\sqrt{x}}$.

हल : $$\int\frac{1}{x-\sqrt{x}}dx = \int\frac{1}{\sqrt{x}(\sqrt{x}-1)}dx$$

$\because$ $\sqrt{x}-1=t$ रखने पर

$\therefore$ $\dfrac{1}{2\sqrt{x}}dx = dt$

$$= 2\int\frac{1}{t}dt = 2\log|t|+C$$

$$= 2\log|\sqrt{x}-1|+C.$$ (t का मान रखने पर) **उत्तर**

प्रश्न 11. $\dfrac{x}{\sqrt{x+4}}, x>0$.

हल : $$\int\frac{x}{\sqrt{x+4}}dx = \int\frac{x+4-4}{\sqrt{x+4}}dx$$

$$= \int\frac{x+4}{\sqrt{x+4}}dx - \int\frac{4}{\sqrt{x+4}}dx$$

$$= \int\sqrt{x+4}\,dx - 4\int(x+4)^{-\frac{1}{2}}dx$$

$$= \frac{2}{3}(x+4)^{\frac{3}{2}} - 4.2(x+4)^{\frac{1}{2}}+C$$

$$= \frac{2}{3}\sqrt{(x+4)}\,(x+4) - 8\sqrt{(x+4)}+C$$

$$= 2\sqrt{(x+4)}\left(\frac{x+4}{3}-4\right)+C$$

$$= \frac{2}{3}\sqrt{x+4}\,(x+4-12)+C$$

$$= \frac{2}{3}\sqrt{x+4}(x-8)+C.$$ **उत्तर**

प्रश्न 12. $(x^3-1)^{\frac{1}{3}} x^5$.

हल : $$\int (x^3-1)^{\frac{1}{3}} x^5 dx$$

$$= \int (x^3-1)^{\frac{1}{3}} x^3 . x^2 dx$$

$\because$ $x^3 - 1 = t$ या $x^3 = 1 + t$ रखने पर

$\therefore$ $3x^2\, dx = dt$

$$= \frac{1}{3}\int t^{\frac{1}{3}}(1+t)\, dt$$

$$= \frac{1}{3}\int (t^{\frac{1}{3}} + t^{\frac{4}{3}})\, dt$$

$$= \frac{1}{3}\left[\frac{t^{\frac{1}{3}+1}}{\frac{1}{3}+1} + \frac{t^{\frac{4}{3}+1}}{\frac{4}{3}+1}\right] + C$$

$$= \frac{1}{3}\left[\frac{3}{4}t^{\frac{4}{3}} + \frac{3}{7}t^{\frac{7}{3}}\right] + C$$

$$= \frac{1}{4}t^{\frac{4}{3}} + \frac{1}{7}t^{\frac{7}{3}} + C$$

$$= \frac{1}{4}(x^3-1)^{\frac{4}{3}} + \frac{1}{7}(x^3-1)^{\frac{7}{3}} + C.$$

(t का मान रखने पर) **उत्तर**

प्रश्न 13. $\dfrac{x^2}{(2+3x^3)^3}$.

हल : $$\int \frac{x^2}{(2+3x^3)^3} dx$$

$\because$ $2 + 3x^3 = t$ रखने पर

$\therefore$ $9x^2\, dx = dt$

$$= \frac{1}{9}\int \frac{dt}{t^3} = \frac{1}{9}\int t^{-3} dt$$

$$= \frac{1}{9}\left[\frac{t^{-3+1}}{-3+1}\right] + C$$

$$= -\frac{1}{18}.\frac{1}{t^2} + C$$

$$= -\frac{1}{18(2+3x^3)^2} + C.$$

(t का मान रखने पर) **उत्तर**

प्रश्न 14. $\dfrac{1}{x(\log x)^m}, x > 0, m \neq 1.$

हल : $\displaystyle\int \frac{1}{x(\log x)^m} dx$

$\because$ $\log x = t$ रखने पर

$\therefore$ $\frac{1}{x}dx = dt$

$$= \int \frac{1}{t^m} dt = \int t^{-m} dt = \frac{t^{-m+1}}{-m+1} + C$$

$$= \frac{(\log x)^{1-m}}{1-m} + C.$$ (t का मान रखने पर) **उत्तर**

प्रश्न 15. $\dfrac{x}{9-4x^2}.$

हल : $\displaystyle\int \frac{x}{9-4x^2} dx$

$\because$ $9 - 4x^2 = t$ रखने पर

$\therefore$ $-8x\, dx = dt$

$$= -\frac{1}{8}\int \frac{dt}{t} = -\frac{1}{8}\log|t| + C$$

$$= \frac{1}{8}\log|t|^{-1} + C$$

$$= \frac{1}{8}\log\frac{1}{(9-4x^2)} + C.$$ (t का मान रखने पर) **उत्तर**

प्रश्न 16. $e^{2x+3}.$

हल : $\displaystyle\int e^{2x+3} dx$

$\because$ $2x + 3 = t$ रखने पर

$\therefore$ $2dx = dt$

$$= \frac{1}{2}\int e^t dt = \frac{1}{2}e^t + C$$

$$= \frac{1}{2}e^{2x+3} + C.$$ (t का मान रखने पर) **उत्तर**

प्रश्न 17. $\dfrac{x}{e^{x^2}}.$

हल : $\displaystyle\int \frac{x}{e^{x^2}} dx$

$\because$ $x^2 = t$ रखने पर

$\therefore$ $2x\, dx = dt$

$\therefore$ $$= \frac{1}{2}\int \frac{dt}{e^t} = \frac{1}{2}\int e^{-t} dt = -\frac{1}{2}e^{-t} + C$$

$$= -\frac{e^{-x^2}}{2} + C$$

$$= -\frac{1}{2e^{x^2}} + C.$$ (t का मान रखने पर) **उत्तर**

प्रश्न 18. $\frac{e^{\tan^{-1} x}}{1+x^2}$.

हल : $\int \frac{e^{\tan^{-1} x}}{1+x^2} dx$

$\because$ $\tan^{-1} x = t$ रखने पर

$\therefore$ $\frac{1}{1+x^2} dx = dt$

$$= \int e^t dt = e^t + C = e^{\tan^{-1} x} + C.$$

(t का मान रखने पर) **उत्तर**

प्रश्न 19. $\frac{e^{2x} - 1}{e^{2x} + 1}$.

हल : $\int \frac{e^{2x} - 1}{e^{2x} + 1} dx = \int \frac{e^x (e^x - e^{-x})}{e^x (e^x + e^{-x})} dt$

$$= \int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$$

$\because$ $e^x + e^{-x} = t$ रखने पर

$\therefore$ $(e^x - e^{-x})\, dx = dt$

$$= \int \frac{dt}{t} = \log |t| + C$$

$$= \log (e^x + e^{-x}) + C.$$ (t का मान रखने पर) **उत्तर**

प्रश्न 20. $\frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}$.

हल : $\int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} dx$

$\because$ $e^{2x} + e^{-2x} = t$ रखने पर

$\therefore$ $(2e^{2x} - 2e^{-2x})\, dx = dt$

या $2(e^{2x} - e^{-2x})\, dx = dt$

$$= \frac{1}{2} \int \frac{dt}{t} = \frac{1}{2} \log |t| + C$$

$$= \frac{1}{2} \log |e^{2x} + e^{-2x}| + C.$$ (t का मान रखने पर) **उत्तर**

प्रश्न 21. $\tan^2 (2x - 3)$.

हल : $\int \tan^2 (2x - 3)\, dx$

$$= \int [\sec^2(2x-3) - 1]\, dx$$

$$= \int \sec^2(2x-3)\, dx - \int dx$$

$\because$ $2x - 3 = t$ रखने पर

$\therefore$ $2dx = dt$

$$= \frac{1}{2}\int \sec^2 t\, dt - \int dx = \frac{1}{2}\tan t - x + C$$

$$= \frac{1}{2}\tan(2x-3) - x + C.$$ (t का मान रखने पर) उत्तर

प्रश्न 22. $\sec^2(7 - 4x)$.

हल : $\int \sec^2(7-4x)\, dx$

$\because$ $7 - 4x = t$ रखने पर

$\therefore$ $-4\, dx = dt$

$$= -\frac{1}{4}\int \sec^2 t\, dt = -\frac{1}{4}\tan t + C$$

$$= -\frac{1}{4}\tan(7-4x) + C.$$ (t का मान रखने पर) उत्तर

प्रश्न 23. $\dfrac{\sin^{-1} x}{\sqrt{1-x^2}}$.

हल : $\dfrac{\sin^{-1} x}{\sqrt{1-x^2}}\, dx$

$\because$ $\sin^{-1} x = t$ रखने पर

$\therefore$ $\dfrac{1}{\sqrt{1-x^2}}\, dx = dt$

$$= \int t\, dt = \frac{t^2}{2} + C$$

$$= \frac{(\sin^{-1} x)^2}{2} + C.$$ उत्तर

प्रश्न 24. $\dfrac{2\cos x - 3\sin x}{6\cos x + 4\sin x}$.

हल : $\displaystyle\int \frac{2\cos x - 3\sin x}{6\cos x + 4\sin x}\, dx$

$$= \frac{1}{2}\int \frac{2\cos x - 3\sin x}{3\cos x + 2\sin x}\, dx$$

$\because$ $2\sin x + 3\cos x = t$ रखने पर

$\therefore$ $(2\cos x - 3\sin x)\, dx = dt$

$$= \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\log|t| + C$$

$$= \frac{1}{2}\log|2\sin x + 3\cos x| + C.$$ उत्तर

प्रश्न 25. $\dfrac{1}{\cos^2 x(1-\tan x)^2}$.

हल : $$\int \frac{1}{\cos^2 x(1-\tan x)^2}dx = \int \frac{\sec^2 x}{(1-\tan x)^2}dx$$

$\because$ $1 - \tan x = t$ रखने पर

$\therefore -\sec^2 x\, dx = dt$

$$= -\int \frac{1}{t^2}dt = -\int t^{-2}dt = \frac{-t^{-2+1}}{-2+1} + C$$

$$= -\frac{t^{-1}}{-1} + C = \frac{1}{t} + C$$

$$= -\frac{1}{1-\tan x} + C.$$ उत्तर

प्रश्न 26. $\dfrac{\cos\sqrt{x}}{\sqrt{x}}$.

हल : $$\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx$$

$\because$ $\sqrt{x} = t$ रखने पर

$\therefore$ $\dfrac{1}{2\sqrt{x}}dx = dt$

$\therefore$ $$= 2\int \cos t\, dt = 2 \sin t + C$$

$$= 2\sin\sqrt{x} + C.$$ उत्तर

प्रश्न 27. $\sqrt{\sin 2x}\cos 2x$.

हल : $$\int \sqrt{\sin 2x}\cos 2x\, dx$$

$\because$ $\sin 2x = t$ रखने पर

$\therefore$ $2\cos 2x\, dx = dt$

$$= \frac{1}{2}\int \sqrt{t}\, dt = \frac{1}{2}\times\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$$

$$= \frac{1}{2}\times\frac{2}{3}t^{\frac{3}{2}} + C = \frac{1}{3}(\sin 2x)^{\frac{3}{2}} + C.$$ उत्तर

प्रश्न 28. $\dfrac{\cos x}{\sqrt{1+\sin x}}$.

हल : $$\int \frac{\cos x}{\sqrt{1+\sin x}}dx$$

$\because$ $\sin x = t$ रखने पर

$\therefore$ $\cos x\, dx = dt$

$$= \int \frac{dt}{\sqrt{1+t}} = \frac{(1+t)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$$

$$= 2(1+t)^{\frac{1}{2}} + C$$

$$= 2\sqrt{(1+\sin x)} + C.$$ उत्तर

प्रश्न 29. cot x log sin x.

हल : $\int \cot x \log \sin x \, dx$

$\because$ $\log \sin x = t$ रखने पर,

$\therefore$ $\cot x \, dx = dt$

$$= \int t \, dt = \frac{t^2}{2} + C$$

$$= \frac{1}{2}(\log \sin x)^2 + C.$$ उत्तर

प्रश्न 30. $\dfrac{\sin x}{1+\cos x}$.

हल : $\int \frac{\sin x}{1+\cos x} dx$

$\because$ $1 + \cos x = t$ रखने पर

$\therefore$ $-\sin x \, dx = dt$

$$= -\int \frac{1}{t} dt = -\log|t| + C$$

$$= -\log|1+\cos x| + C$$

$$= \log\left|\frac{1}{1+\cos x}\right| + C.$$ उत्तर

प्रश्न 31. $\dfrac{\sin x}{(1+\cos x)^2}$.

हल : $\int \frac{\sin x}{(1+\cos x)^2} dx$

$\because$ $1 + \cos x = t$ रखने पर,

$\therefore$ $-\sin x \, dx = dt$

$$= -\int \frac{dt}{t^2} = -\frac{t^{-2+1}}{-2+1} + C = \frac{1}{t} + C$$

$$= \frac{1}{1+\cos x} + C.$$ उत्तर

प्रश्न 32. $\dfrac{1}{1+\cot x}$.

हल : $\int \frac{1}{1+\cot x} dx$

$$= \int \frac{1}{1+\frac{\cos x}{\sin x}} dx$$

$$= \int \frac{1}{\frac{\sin x + \cos x}{\sin x}} dx$$

$$= \int \frac{\sin x}{\sin x + \cos x} dx$$

$$= \frac{1}{2}\int \left(\frac{2\sin x}{\sin x + \cos x}\right) dx$$

$$= \frac{1}{2}\int \frac{(\sin x + \cos x) - (\cos x - \sin x)}{\sin x + \cos x} dx$$

$$= \int \frac{1}{2} dx - \frac{1}{2}\int \frac{\cos x - \sin x}{\sin x + \cos x} dx$$

$\because$ $\sin x + \cos x = t$ रखने पर,

$\therefore$ $(\cos x - \sin x)\, dx = dt$

$$= \frac{1}{2}\int dx - \frac{1}{2}\int \frac{1}{t} dt$$

$$= \frac{1}{2}x - \frac{1}{2}\log |t| + C$$

$$= \frac{1}{2}x - \frac{1}{2}\log |\sin x + \cos x| + C.$$ **उत्तर**

प्रश्न 33. $\frac{1}{1-\tan x}$.

हल :

$$\int \frac{1}{1-\tan x} dx = \int \frac{1}{1-\frac{\sin x}{\cos x}} dx = \int \frac{1}{\frac{\cos x - \sin x}{\cos x}} dx$$

$$= \int \frac{\cos x}{\cos x - \sin x} dx = \frac{1}{2}\int \frac{2\cos x}{\cos x - \sin x} dx$$

$$= \frac{1}{2}\int \frac{\cos x - \sin x + \cos x + \sin x}{\cos x - \sin x} dx$$

$$= \frac{1}{2}\int dx - \frac{1}{2}\int \frac{-\cos x - \sin x}{\cos x - \sin x} dx$$

$$= \frac{1}{2}\int dx - \frac{1}{2}\int \frac{-\sin x - \cos x}{\cos x - \sin x} dx$$

$\therefore$ $\cos x - \sin x = t$ रखने पर,

$\therefore$ $(-\sin x - \cos x)\, dx = dt$

$$= \frac{1}{2}\int dx - \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}x - \frac{1}{2}\log |t| + C$$

$$= \frac{1}{2}x - \frac{1}{2}\log |\cos x - \sin x| + C.$$ **उत्तर**

प्रश्न 34. $\dfrac{\sqrt{\tan x}}{\sin x \cos x}$.

हल :
$$\int \frac{\sqrt{\tan x}}{\sin x \cos x} dx = \int \frac{\sqrt{\tan x}}{\frac{\sin x}{\cos x} \cos^2 x} dx$$
$$= \int \frac{\sqrt{\tan x}}{\tan x} \sec^2 x \, dx = \int \frac{\sec^2 x}{\sqrt{\tan x}} dx$$

$\because$ $\tan x = t$ रखने पर
$\therefore$ $\sec^2 x \, dx = dt$

$$= \int \frac{dt}{\sqrt{t}} = \int t^{-\frac{1}{2}} dt = \int \frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$$
$$= 2\sqrt{t} + C = 2\sqrt{\tan x} + C.$$ उत्तर

प्रश्न 35. $\dfrac{(1 + \log x)^2}{x}$.

हल :
$$\int \frac{(1 + \log x)^2}{x} dx$$

$\because$ $1 + \log x = t$ रखने पर,

$\therefore$ $$\frac{1}{x} dx = dt$$

$$= \int t^2 dt = \frac{t^3}{3} + C = \frac{1}{3}(1 + \log x)^3 + C.$$ उत्तर

प्रश्न 36. $\dfrac{(x + 1)(x + \log x)^2}{x}$.

हल :
$$\int \frac{(x + 1)(x + \log x)^2}{x} dx = \int \left(1 + \frac{1}{x}\right)(x + \log x)^2 dx$$

$\because$ $x + \log x = t$ रखने पर,

$\therefore$ $\left(1 + \frac{1}{x}\right) dx = dt$

$$= \int t^2 dt = \frac{t^3}{3} + C = \frac{1}{3}(x + \log x)^3 + C.$$ उत्तर

प्रश्न 37. $\dfrac{x^3 \sin (\tan^{-1} x^4)}{1 + x^8}$.

हल :
$$\int \frac{x^3 \sin (\tan^{-1} x^4)}{1 + x^8} dx$$

$\because$ $\tan^{-1} x^4 = t$ रखने पर,

$$\therefore \qquad \frac{1}{1+x^8}\frac{d}{dx}x^4 = dt$$

या $$\frac{4x^3}{1+x^8}dx = dt$$

या $$\frac{x^3}{1+x^8}dx = \frac{1}{4}dt$$

$$= \frac{1}{4}\int \sin t \; dt$$

$$= \frac{1}{4}(-\cos t) + C$$

$$= -\frac{1}{4}\cos\ (\tan^{-1} x^4) + C.$$ उत्तर

प्रश्न 38 एवं 39 में सही उत्तर का चयन कीजिए—

प्रश्न 38. $\int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x} dx$ **बराबर है—**

(A) $10^x - x^{10} + C$ **(B)** $10^x + x^{10} + C$

(C) $(10^x - x^{10})^{-1} + C$ **(D)** $\log(10^x + x^{10}) + C$

हल : $$\int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x} dx$$

$\because x^{10} + 10^x = t$ रखने पर

$\therefore (10x^9 + 10^x \log_e 10)\, dx = dt$

$$= \int \frac{dt}{t} = \log|t| + C$$

$$= \log|x^{10} + 10^x| + C$$

अतः विकल्प (D) सही है। उत्तर

प्रश्न 39. $\int \frac{dx}{\sin^2 x \cos^2 x} dx$ **बराबर है—**

(A) $\tan x + \cot x + C$ **(B)** $\tan x - \cot x + C$

(C) $\tan x \cot x + C$ **(D)** $\tan x - \cot 2x + C$

हल : $$\int \frac{dx}{\sin^2 x \cos^2 x} dx = \int \frac{(\sin^2 x + \cos^2 x)}{\sin^2 x \cos^2 x} dx$$

$$= \int \frac{\sin^2 x}{\sin^2 x \cos^2 x} dx + \int \frac{\cos^2 x\, dx}{\sin^2 x \cos^2 x}$$

$$= \int \sec^2 x\, dx + \int \operatorname{cosec}^2 x\, dx$$

$$= \tan x - \cot x + C.$$

अतः विकल्प (B) सही है। उत्तर

प्रश्नावली 7·3

1 से 22 तक के प्रश्नों में प्रत्येक फलन का समाकलन ज्ञात कीजिए—

प्रश्न 1. $\sin^2 (2x + 5)$.

हल : $\int \sin^2(2x+5)\, dx$ $\left(\because \sin^2 A = \dfrac{1-\cos 2A}{2}\right)$

$$= \frac{1}{2}\int [1-\cos 2(2x+5)]\, dx$$

$$= \frac{1}{2}\int [1-\cos (4x+10)]\, dx$$

यहाँ $4x + 10 = t$ रखने पर

$4dx = dt$

$$= \frac{1}{2}\int 1\, dx - \frac{1}{2}.\frac{1}{4}\int \cos t\, dt$$

$$= \frac{1}{2}x - \frac{1}{8}\sin t + C$$

$$= \frac{1}{2}x - \frac{1}{8}\sin (4x+10) + C.$$ उत्तर

प्रश्न 2. $\sin 3x \cos 4x$.

हल : $$\int \sin 3x \cos 4x\, dx = \frac{1}{2}\int 2\cos 4x \sin 3x\, dx$$

$$= \frac{1}{2}\int \sin 7x - \sin x\, dx$$

$[\because 2 \cos A \sin B = \sin (A + B) - \sin (A - B)]$

$$= \frac{1}{2}\int \sin\, 7x dx - \frac{1}{2}\int \sin\, x dx$$

$$= \frac{1}{2}\left(-\frac{\cos 7x}{7}\right) + \frac{1}{2}(\cos x) + C$$

$$= -\frac{1}{14}\cos 7x + \frac{1}{2}\cos x + C.$$ उत्तर

प्रश्न 3. $\cos 2x \cos 4x \cos 6x$.

हल : $$\int \cos 2x \cos 4x \cos 6x\, dx = \frac{1}{2}\int (2\cos 4x \cos 2x)\cos 6x\, dx$$

$$= \frac{1}{2}\int [\cos 6x + \cos 2x]\cos 6x\, dx$$

$[\because 2 \cos A \cos B = \cos (A + B) - \cos (A - B)]$

$$= \frac{1}{4}\int (2\cos^2 6x + 2\cos 6x \cos 2x)\, dx$$

$$= \frac{1}{4}\int [1 + \cos 12x + \cos 8x + \cos 4x]\, dx$$

$$= \frac{1}{4}\left[\int 1\,dx + \int \cos 12x\,dx + \int \cos 8x\,dx + \int \cos 4x\,dx\right]$$

$$= \frac{1}{4}\left[x + \frac{\sin 12x}{12} + \frac{\sin 8x}{8} + \frac{\sin 4x}{4}\right] + C.$$ उत्तर

प्रश्न 4. $\sin^3 (2x + 1)$.

हल :
$$\int \sin^3(2x+1)\,dx = \int \sin^2(2x+1).\sin(2x+1)\,dx$$

$$= \int [1-\cos^2(2x+1)].\sin(2x+1)\,dx$$

मान लीजिए $\cos (2x + 1) = t$

$$-\sin(2x+1)2dx = dt$$

या
$$\sin(2x+1)\,dx = \frac{dt}{2}$$

$$= \frac{-1}{2}\int (1-t^2)\,dt$$

$$= \frac{-1}{2}\left[\int 1\,dt - \int t^2 dt\right]$$

$$= \frac{-1}{2}\left[t - \frac{t^3}{3}\right] + C$$

$$= \frac{-t}{2} + \frac{t^3}{6} + C$$

$$= -\frac{\cos(2x+1)}{2} + \frac{\cos^3(2x+1)}{6} + C$$ उत्तर

प्रश्न 5. $\sin^3 x \cos^3 x$.

हल :
$$\int \sin^3 x \cos^3 x\,dx = \int \sin^2 x \cos^3 x \sin x\,dx$$

$$= \int (1-\cos^2 x)\cos^3 x \sin x\,dx$$

$\because$ $\cos x = t$ रखने पर

$\therefore$ $-\sin x\,dx = dt$

$$= -\int (1-t^2)t^3 dt = -\int (t^3 - t^5)dt$$

$$= -\frac{t^4}{4} + \frac{t^6}{6} + C$$

$$= \frac{1}{6}\cos^6 x - \frac{1}{4}\cos^4 x + C.$$ उत्तर

प्रश्न 6. $\sin x \sin 2x. \sin 3x$.

हल :
$$\int (\sin x \sin 2x \sin 3x)\,dx$$

$$= \frac{1}{2}\int (2\sin x \sin 2x)\sin 3x\,dx$$

$$= \frac{1}{2}\int (\cos x - \cos 3x)\sin 3x\ dx$$

[∵ 2 sin A sin B = cos (A – B) – cos (A + B)]

$$= \frac{1}{4}\int [2\sin 3x\cos x - 2\sin 3x\cos 3x]\ dx$$

$$= \frac{1}{4}\int [\sin 4x + \sin 2x - \sin 6x]\ dx$$

[∵ 2 sin A cos B = sin (A + B) + sin (A – B)]

$$= -\frac{1}{4}\left[\frac{\cos 4x}{4} + \frac{\cos 2x}{2} - \frac{\cos 6x}{6}\right] + C$$

$$= \frac{1}{4}\left(\frac{\cos 6x}{6} - \frac{\cos 4x}{4} - \frac{\cos 2x}{2}\right) + C.$$ उत्तर

प्रश्न 7. sin 4x sin 8x.

हल :

$$\int \sin 4x \sin 8x\ dx = \frac{1}{2}\int (2\sin 8x\sin 4x)\ dx$$

[∵ 2 sin A sin B = cos (A – B) – cos (A + B)]

$$= \frac{1}{2}\int [\cos 4x - \cos 12x]\ dx$$

$$= \frac{1}{2}\left[\int \cos 4x\, dx - \int \cos 12x\ dx\right]$$

$$= \frac{1}{2}\left[\frac{1}{4}\sin 4x - \frac{1}{12}\sin 12x\right] + C.$$ उत्तर

प्रश्न 8. $\frac{1-\cos x}{1+\cos x}$.

हल :

$$\int \frac{1-\cos x}{1+\cos x}\ dx = \int \frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}\ dx$$

$$\left[\because 1 - \cos A = 2\sin^2\frac{A}{2},\ 1 + \cos A = 2\cos^2\frac{A}{2}\right]$$

$$= \int \tan^2\frac{x}{2}dx = \int\left(\sec^2\frac{x}{2} - 1\right)dx$$

$$= \int \sec^2\frac{x}{2}dx - \int 1.dx$$

$$= \frac{\tan\frac{x}{2}}{\frac{1}{2}} - x + C$$

$$= 2\tan\frac{x}{2} - x + C.$$ उत्तर

प्रश्न 9. $\dfrac{\cos x}{1+\cos x}$.

हल :

$$\int \frac{\cos x}{1+\cos x}dx = \int \frac{1+\cos x-1}{1+\cos x}dx.$$

$$= \int \frac{1+\cos x}{1+\cos x}dx - \int \frac{1}{1+\cos x}dx$$

$$= \int\left(1-\frac{1}{1+\cos x}\right)dx = \int\left(1-\frac{1}{2\cos^2\frac{x}{2}}\right)dx$$

$[\because 1 + \cos 2A = 2\cos^2 A]$

$$= \int\left(1-\frac{1}{2}\sec^2\frac{x}{2}\right)dx = \int 1.dx - \frac{1}{2}\int \sec^2\frac{x}{2}dx$$

$$= x-\frac{1}{2}.\frac{\tan\frac{x}{2}}{\frac{1}{2}}+C = x-\tan\frac{x}{2}+C.$$ **उत्तर**

प्रश्न 10. $\sin^4 x$.

हल :

$$\int \sin^4 x\, dx = \int (\sin^2 x)^2 dx$$

$$= \int\left(\frac{1-\cos 2x}{2}\right)^2 dx \qquad \left[\because \sin^2 x = \frac{1-\cos 2x}{2}\right]$$

$$= \frac{1}{4}\int (1-2\cos 2x+\cos^2 2x)\, dx$$

$$= \frac{1}{4}\int\left(1-2\cos 2x+\frac{1+\cos 4x}{2}\right)dx$$

$$\left[\because \cos^2 A = \frac{1+\cos 2A}{2}\right]$$

$$= \frac{1}{8}\int (2-4\cos 2x+\cos 4x+1)\, dx$$

$$= \frac{1}{8}\int (3-4\cos 2x+\cos 4x)\, d$$

$$= \frac{1}{8}\left[\int 3dx - 4\int \cos 2x\, dx + \int \cos 4x\, dx\right]$$

$$= \frac{1}{8}\left[3x-4.\frac{\sin 2x}{2}+\frac{\sin 4x}{4}\right]+C$$

$$= \frac{3}{8}x-\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+C.$$ **उत्तर**

प्रश्न 11. $\cos^4 2x$.

हल :

$$\int \cos^4 2x\, dx = \int (\cos^2 2x)^2 dx$$

$$= \int \left(\frac{1+\cos 4x}{2}\right)^2 dx \qquad \left(\because \cos^2 A = \frac{1+\cos 2A}{2}\right)$$

$$= \frac{1}{4}\int (1 + 2\cos 4x + \cos^2 4x)\, dx$$

$$= \frac{1}{4}\int \left(1 + 2\cos 4x + \frac{1+\cos 8x}{2}\right) dx$$

$$= \frac{1}{4}\int \frac{(2 + 4\cos 4x + 1 + \cos 8x}{2}\, dx$$

$$= \frac{1}{8}\int (3 + 4\cos 4x + \cos 8x)\, dx$$

$$= \frac{1}{8}\left[\int 3\,dx + 4\int \cos\, 4x\,dx + \int \cos\, 8x\,dx\right]$$

$$= \frac{1}{8}\left(3x + \frac{4\sin 4x}{4} + \frac{\sin 8x}{8}\right) + C$$

$$= \frac{3}{8}x + \frac{1}{8}\sin 4x + \frac{1}{64}\sin 8x + C$$ उत्तर

प्रश्न 12. $\dfrac{\sin^2 x}{1+\cos x}$.

हल :

$$\int \frac{\sin^2 x}{1+\cos x} dx = \int \frac{1-\cos^2 x}{1+\cos x} dx$$

$$= \int \frac{(1-\cos x)(1+\cos x)}{(1+\cos x)} dx$$

$$= \int (1-\cos x)\, dx = x - \sin x + C.$$ उत्तर

प्रश्न 13. $\dfrac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha}$.

हल :

$$\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha} dx = \int \frac{(2\cos^2 x - 1) - (2\cos^2 \alpha - 1)}{\cos x - \cos\alpha} dx$$

$$= \int \frac{2(\cos^2 x - \cos^2 \alpha)}{\cos x - \cos\alpha} dx$$

$$= 2\int \frac{(\cos x + \cos\alpha)(\cos x - \cos\alpha)}{\cos x - \cos\alpha} dx$$

$$= 2\int (\cos x + \cos\alpha)\, dx$$

$$= 2\left[\int \cos\, x\,dx + \cos\,\alpha \int 1\, dx\right]$$

$$= 2(\sin x + x\cos\alpha) + C.$$ उत्तर

प्रश्न 14. $\dfrac{\cos x - \sin x}{1+\sin 2x}$.

हल : माना

$$I = \int \frac{\cos x - \sin x}{1+\sin 2x} dx$$

∵ $1 = \cos^2 x + \sin^2 x$

और $\sin 2x = 2 \cos x \sin x$

$$= \int \frac{\cos x - \sin x}{\cos^2 x + \sin^2 x + 2\cos x \sin x} dx$$

$$= \int \frac{\cos x - \sin x}{(\cos x + \sin x)^2} dx$$

माना $\cos x + \sin x = t$ रखने पर

$(-\sin x + \cos x)\, dx = dt$

∴ $$I = \int \frac{dt}{t^2} = \int t^{-2} dt = \frac{t^{-2+1}}{-2+1} + C$$

$$= -\frac{1}{t} + C = -\frac{1}{\cos x + \sin x} + C.$$ **उत्तर**

प्रश्न 15. $\tan^3 2x . \sec 2x$.

हल : $$\int \tan^3 2x \sec 2x\, dx = \int \tan^2 2x . \sec 2x \tan 2x\, dx$$

$$= \int (\sec^2 2x - 1) \sec 2x \tan 2x\, dx$$

∵ $\sec 2x = t$ रखने पर

∴ $2 \sec 2x \tan 2x\, dt = dt$

या $\sec 2x \tan 2x = \frac{1}{2} dt$

$$= \frac{1}{2} \int (t^2 - 1)\, dt$$

$$= \frac{1}{2}\left(\frac{t^3}{3} - t\right) + C$$

$$= \frac{1}{2}\left(\frac{1}{3}\sec^3 2x - \sec 2x\right) + C$$

$$= \frac{1}{6}\sec^3 2x - \frac{1}{2}\sec 2x + C.$$ **उत्तर**

प्रश्न 16. $\tan^4 x$.

हल : $$\int \tan^4 x\, dx = \int \tan^2 x . \tan^2 x\, dx$$

$$= \int \tan^2 x (\sec^2 x - 1)\, dx$$

$$= \int \tan^2 x \sec^2 x\, dx - \int \tan^2 x\, dx$$

$$= \int \tan^2 x \sec^2 x\, dx - \int (\sec^2 x - 1)\, dx$$

∵ $\tan x = t$ रखने पर

∴ $\sec^2 x\, dx = dt$

$$= \int t^2 dt - \int 1 dt + \int 1 . dx$$

$$= \frac{t^3}{3} - t + x + C$$

$$= \frac{1}{3}\tan^3 x - \tan x + x + C.$$ उत्तर

प्रश्न 17. $\dfrac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}$.

हल :

$$\int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} dx = \int \frac{\sin^3 x}{\sin^2 x \cos^2 x} dx + \int \frac{\cos^3 x}{\sin^2 x \cos^2 x} dx$$

$$= \int \frac{\sin x}{\cos^2 x} dx + \int \frac{\cos x}{\sin^2 x} dx$$

$$= \int \sec x \tan x \, dx + \int \operatorname{cosec} x \cot x \, dx$$

$$= \sec x - \operatorname{cosec} x + C.$$ उत्तर

प्रश्न 18. $\dfrac{\cos 2x + 2\sin^2 x}{\cos^2 x}$.

हल :

$$\int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x} dx = \int \frac{\cos^2 x - \sin^2 x + 2\sin^2 x}{\cos^2 x} dx$$

$[\because \cos 2x = \cos^2 x - \sin^2 x]$

$$= \int \frac{\cos^2 x + \sin^2 x}{\cos^2 x} dx$$

$$= \int \frac{1}{\cos^2 x} dx = \int \sec^2 x \, dx$$

$$= \tan x + C.$$ उत्तर

प्रश्न 19. $\dfrac{1}{\sin x \cos^3 x}$.

हल :

$$\int \frac{1}{\sin x \cos^3 x} dx = \int \frac{\sin^2 x + \cos^2 x}{\sin x \cos^3 x} dx$$

$$= \int \left(\frac{\sin^2 x}{\sin x \cos^3 x} + \frac{\cos^2 x}{\sin x \cos^3 x} \right) dx$$

$$= \int \left(\frac{\tan x}{\cos^2 x} + \frac{1}{\tan x . \cos^2 x} \right) dx$$

$$= \int \left(\tan x \sec^2 x + \frac{1}{\tan x} \sec^2 x \right) dx$$

अब $\tan x = t$ रखने पर

$$\sec^2 x \, dx = dt$$

$$\therefore \quad I = \int \left(t + \frac{1}{t} \right) dt = \frac{t^2}{2} + \log |x| + C$$

$$= \frac{1}{2}\tan^2 x + \log|\tan x| + C$$

$$= \log|\tan x| + \frac{1}{2}\tan^2 x + C.$$ उत्तर

प्रश्न 20. $\dfrac{\cos 2x}{(\cos x + \sin x)^2}$.

हल :
$$\int \frac{\cos 2x}{(\cos x + \sin x)^2} dx = \int \frac{\cos^2 x - \sin^2 x}{(\cos x + \sin x)^2} dx$$

$$= \int \frac{(\cos x - \sin x)(\cos x + \sin x)}{(\cos x + \sin x)^2} dx$$

$$= \int \frac{\cos x - \sin x}{\cos x + \sin x} dx$$

$\because$ $\cos x + \sin x = t$ रखने पर,
$\therefore$ $(-\sin x + \cos x)\, dx = dt$

$$= \int \frac{dt}{t} = \log|t| + C$$

$$= \log|\cos x + \sin x| + C.$$ उत्तर

प्रश्न 21. $\sin^{-1}(\cos x)$.

हल :
$$\int \sin^{-1}(\cos x)\, dx = \int \sin^{-1}\left(\sin\left(\frac{\pi}{2} - x\right)\right) dx$$

$$= \int \left(\frac{\pi}{2} - x\right) dx = \frac{\pi}{2}x - \frac{x^2}{2} + C.$$ उत्तर

प्रश्न 22. $\dfrac{1}{\cos(x-a)\cos(x-b)}$.

हल :
$$\int \frac{1}{\cos(x-a)\cos(x-b)} dx$$

$$= \frac{1}{\sin(a-b)} \int \frac{\sin(a-b)}{\cos(x-a)\cos(x-b)} dx$$

$$= \frac{1}{\sin(a-b)} \cdot \int \frac{\sin[(x-b)-(x-a)]}{\cos(x-a)\cos(x-b)} dx$$

$$= \frac{1}{\sin(a-b)} \int \frac{\sin(x-b)\cos(x-a) - \cos(x-b)\sin(x-a)}{\cos(x-a)\cos(x-b)} dx$$

$[\because \sin(A-B) = \sin A \cos B - \cos A \sin B]$

$$= \frac{1}{\sin(a-b)} \int [\tan(x-b) - \tan(x-a)]\, dx$$

$$= \frac{1}{\sin(a-b)} [-\log|\cos(x-b)| + \log|\cos(x-a)|] + C$$

$$= \frac{1}{\sin(a-b)} \log\left|\frac{\cos(x-a)}{\cos(x-b)}\right| + C.$$ उत्तर

प्रश्न 23 एवं 24 में सही उत्तर का चयन कीजिए—

प्रश्न 23. $\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx$ **बराबर है—**

(A) $\tan x + \cot x + C$ (B) $\tan x + \text{cosec } x + C$

(C) $-\tan x + \cot x + C$ (D) $\tan x + \sec x + C$

हल :

$$\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx = \int \frac{\sin^2 x \, dx}{\sin^2 x \cos^2 x} - \int \frac{\cos^2 x \, dx}{\sin^2 x \cos^2 x}$$

$$= \int \frac{dx}{\cos^2 x} - \int \frac{dx}{\sin^2 x}$$

$$= \int \sec^2 x \, dx - \int \text{cosec}^2 x \, dx$$

$$= \tan x + \cot x + C.$$

अत: विकल्प (A) सही है। **उत्तर**

प्रश्न 24. $\int \frac{e^x (1+x)}{\cos^2 (e^x x)} dx$ **बराबर है—**

(A) $-\cot (e.x^x) + C$ (B) $\tan (x e^x) + C$

(C) $\tan (e^x) + C$ (D) $\cot (e^x) + C$

हल :

$$\int \frac{e^x (1+x)}{\cos^2 (e^x x)} dx = \int \frac{(e^x + xe^x) dx}{\cos^2 (e^x x)}$$

मान लीजिए $xe^x = t$

$\Rightarrow (e^x + xe^x) dx = dt$

$$= \int \frac{dt}{\cos^2 t} = \int \sec^2 t \, dt$$

$$= \tan t + C$$

$$= \tan (xe^x) + C$$

अत: विकल्प (B) सही है। **उत्तर**

प्रश्नावली 7·4

प्रश्न 1 से 23 तक के फलनों का समाकलन कीजिए—

प्रश्न 1. $\frac{3x^2}{x^6 + 1}$.

हल :

$$\int \frac{3x^2}{x^6 + 1} dx$$

$\because x^3 = t$ रखने पर,

$\therefore$ $3x^2 \, dx = dt$

$$= \int \frac{dt}{t^2 + 1} = \tan^{-1} t + C = \tan^{-1} x^3 + C.$$ **उत्तर**

प्रश्न 2. $\frac{1}{\sqrt{1 + 4x^2}}$.

हल : $$\int \frac{1}{\sqrt{1+4x^2}}dx = \int \frac{1}{\sqrt{1+(2x)^2}}dx$$

यहाँ $2x = \tan\theta$ लेने पर,
$\therefore 2dx = \sec^2\theta\, d\theta$

$$= \frac{1}{2}\int \frac{1}{\sqrt{1+\tan^2\theta}}\sec^2\theta\, d\theta$$

$$= \frac{1}{2}\int \frac{\sec^2\theta}{\sec\theta}d\theta$$

$$= \frac{1}{2}\int \sec\theta\, d\theta = \frac{1}{2}\log|\sec\theta\tan\theta| + C$$

$$= \frac{1}{2}\log\left|\sqrt{1+4x^2}+2x\right| + C \quad [\because \sec\theta = \sqrt{1+4x^2}]$$

$$= \frac{1}{2}\log\left|2x+\sqrt{1+4x^2}\right| + C.$$ **उत्तर**

प्रश्न 3. $\frac{1}{\sqrt{(2-x)^2+1}}$.

हल : $$\int \frac{1}{\sqrt{(2-x)^2+1}}$$

यहाँ $2 - x = t$ रखने पर
$\therefore -dx = dt$

$$= -\int \frac{dx}{\sqrt{t^2+1}} = -\log\left|t+\sqrt{t^2+1}\right| + C$$

$$= -\log\left|(2-x)+\sqrt{(2-x)^2+1}\right| + C$$

$$= \log\left|\frac{1}{(2-x)+\sqrt{(2-x)^2+1}}\right| + C$$

$$= \log\left|\frac{1}{2-x+\sqrt{x^2-4x+5}}\right| + C.$$ **उत्तर**

प्रश्न 4. $\frac{1}{\sqrt{9-25x^2}}$.

हल : $$\int \frac{dx}{\sqrt{9-25x^2}} = \frac{1}{5}\int \frac{dx}{\sqrt{\frac{9}{25}-x^2}} = \frac{1}{5}\int \frac{dx}{\sqrt{\left(\frac{3}{5}\right)^2-x^2}}$$

$$\left[\because \int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a}\right]$$

$$= \frac{1}{5}\sin^{-1}\left[\frac{x}{\frac{3}{5}}\right] + C = \frac{1}{5}\sin^{-1}\left(\frac{5x}{3}\right) + C.$$ **उत्तर**

प्रश्न 5. $\dfrac{3x}{1+2x^4}$.

हल :
$$\int \frac{3x\,dx}{1+2x^4} = \int \frac{3x\,dx}{1+2(x^2)^2}$$

$\because x^2 = t$ रखने पर,
$\therefore 2x\,dx = dt$

$$= \frac{3}{2}.\int \frac{2x}{1+2(x^2)^2}dx = \frac{3}{2}\int \frac{dt}{1+2t^2}$$
$$= \frac{3}{2}\int \frac{dt}{\frac{1}{2}+t^2} = \frac{3}{2}\int \frac{dt}{\left(\frac{1}{\sqrt{2}}\right)^2+t^2}$$
$$= \frac{3}{4}\sqrt{2}\tan^{-1}\left(\frac{t}{\frac{1}{\sqrt{2}}}\right)+C$$
$$\left[\because \int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$
$$= \frac{3}{2\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{2}t}{1}\right)+C$$
$$= \frac{3}{2\sqrt{2}}\tan^{-1}\left(\sqrt{2}x^2\right)+C.$$ उत्तर

प्रश्न 6. $\dfrac{x^2}{1-x^6}$.

हल :
$$\int \frac{x^2}{1-x^6}dx = \int \frac{x^2dx}{1-(x^3)^2}$$

$\because x^3 = t$ रखने पर,
$\therefore 3x^2\,dx = dt$

$$= \frac{1}{3}\int \frac{dt}{1-t^2} = \frac{1}{3}.\frac{1}{2}\log\frac{1+t}{1-t}+C$$
$$= \frac{1}{6}\log\left(\frac{1+x^3}{1-x^3}\right)+C.$$ उत्तर

प्रश्न 7. $\dfrac{x-1}{\sqrt{x^2-1}}$.

हल :
$$\int \frac{x-1}{\sqrt{x^2-1}}dx = \int \frac{x}{\sqrt{x^2-1}}dx - \int \frac{dx}{\sqrt{x^2-1}}$$
$$= \frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}}dx - \log\left(x+\sqrt{x^2-1}\right)$$

$\because x^2 - 1 = t$ रखने पर
$\therefore 2x\,dx = dt$

$$= \frac{1}{2}\int \frac{dt}{\sqrt{t}} - \log\,(x+\sqrt{x^2-1})$$

$$= \frac{1}{2}\left[\frac{t^{-\frac{1}{2}+1}}{\frac{1}{2}}\right] - \log\,(x+\sqrt{x^2-1}) + C$$

$$= t^{\frac{1}{2}} - \log\,(x+\sqrt{x^2-1}) + C$$

$$= \sqrt{x^2-1} - \log\,(x+\sqrt{x^2-1}) + C.$$ उत्तर

प्रश्न 8. $\dfrac{x^2}{\sqrt{x^6+a^6}}$.

हल : $\displaystyle\int \frac{x^2dx}{\sqrt{x^6+a^6}} = \int \frac{x^2dx}{(x^3)^2+a^6}$

$\because$ $x^3 = t$ रखने पर,

$\therefore$ $3x^2\,dx = dt$

$$= \frac{1}{3}\int \frac{dt}{\sqrt{t^2+a^6}} = \frac{1}{3}\int \frac{dt}{\sqrt{t^2+(a^3)^2}}$$

$$= \frac{1}{3}\log\left|t+\sqrt{t^2+(a^3)^2}\right|$$

$$\left[\because \int \frac{1}{\sqrt{x^2+a^2}}\,dx = \log\,(x+\sqrt{x^2+a^2})\right]$$

$$= \frac{1}{3}\log\left|x^3+\sqrt{x^6+a^6}\right| + C.$$ उत्तर

प्रश्न 9. $\dfrac{\sec^2 x}{\sqrt{\tan^2 x+4}}$.

हल : $\displaystyle\int \frac{\sec^2 x}{\sqrt{\tan^2 x+4}}\,dx$

यहाँ $\tan x = t$ रखने पर,

$\therefore$ $\sec^2 x\,dx = dt$

$$= \int \frac{dt}{\sqrt{t^2+4}} = \int \frac{dt}{\sqrt{t^2+(2)^2}}$$

$$\left[\because \int \frac{dt}{\sqrt{x^2+a^2}}\,dx = \log\,(x+\sqrt{x^2+a^2})\right]$$

$$= \log\left|\,t+\sqrt{t^2+4}\,\right| + C$$

$$= \log\left|\,\tan x+\sqrt{\tan^2 x+4}\,\right| + C.$$ उत्तर

प्रश्न 10. $\dfrac{1}{\sqrt{x^2+2x+2}}$.

हल :
$$\int \frac{1}{\sqrt{x^2+2x+2}}dx = \int \frac{1}{\sqrt{(x+1)^2+1}}dx$$
$$= \log\left| (x+1)+\sqrt{(x+1)^2+1} \right| + C$$
$$\therefore \quad = \log\left| (x+1)+\sqrt{x^2+2x+2} \right| + C.$$
उत्तर

प्रश्न 11. $\dfrac{1}{9x^2+6x+5}$.

हल :
$$\int \frac{1}{9x^2+6x+5}dx = \frac{1}{9}\int \frac{dx}{x^2+\frac{2}{3}x+\frac{5}{9}}$$
$$= \frac{1}{9}\int \frac{dx}{\left(x^2+\frac{2}{3}x+\frac{1}{9}\right)+\frac{5}{9}-\frac{1}{9}}$$
$$= \frac{1}{9}\int \frac{dx}{\left(x+\frac{1}{3}\right)^2+\frac{4}{9}}$$
$$= \frac{1}{9}\int \frac{dx}{\left(x+\frac{1}{3}\right)^2+\left(\frac{2}{3}\right)^2} \quad \left[\because \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$
$$= \frac{1}{9}\cdot\frac{1}{\left(\frac{2}{3}\right)}\tan^{-1}\frac{x+\frac{1}{3}}{\frac{2}{3}}+C$$
$$= \frac{1}{6}\tan^{-1}\left(\frac{3x+1}{2}\right)+C.$$
उत्तर

प्रश्न 12. $\dfrac{1}{\sqrt{7-6x-x^2}}$.

हल :
$$\int \frac{1}{\sqrt{7-6x-x^2}}dx = \int \frac{dx}{\sqrt{7-(x^2+6x)}}$$
$$= \int \frac{dx}{\sqrt{7-(x^2+6x+9)+9}}$$
$$= \int \frac{dx}{\sqrt{16-(x+3)^2}} \quad \left[\because \int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a}\right]$$
$$= \sin^{-1}\left(\frac{x+3}{4}\right)+C.$$
उत्तर

प्रश्न 13. $\dfrac{1}{\sqrt{(x-1)(x-2)}}$.

हल :

$$\int \frac{dx}{\sqrt{(x-1)(x-2)}} = \int \frac{dx}{\sqrt{x^2-3x+2}}$$

$$= \int \frac{dx}{\sqrt{\left(x^2-3x+\frac{9}{4}\right)+2-\frac{9}{4}}}$$

$$= \int \frac{dx}{\sqrt{\left(x-\frac{3}{2}\right)^2-\frac{1}{4}}}$$

$$= \int \frac{dx}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}}$$

$$= \log\left|\left(x-\frac{3}{2}\right)+\sqrt{\left(x-\frac{3}{2}\right)^2-\frac{1}{4}}\right|+C$$

$$= \log\left|\frac{2x-3}{2}+\sqrt{x^2-3x+2}\right|+C$$

$$= \log\left|x-\frac{3}{2}+\sqrt{x^2-3x+2}\right|+C.$$

उत्तर

प्रश्न 14. $\dfrac{1}{\sqrt{8+3x-x^2}}$.

हल :

$$\int \frac{dx}{\sqrt{8+3x-x^2}} = \int \frac{dx}{\sqrt{8-(x^2-3x)}}$$

$$= \int \frac{dx}{\sqrt{8-\left(x^2-3x+\frac{9}{4}\right)+\frac{9}{4}}}$$

$$= \int \frac{dx}{\sqrt{\frac{41}{4}-\left(x-\frac{3}{2}\right)^2}} = \int \frac{dx}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}}$$

$$\left[\because \int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a}\right]$$

$$= \sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{41}}{2}}\right)+C = \sin^{-1}\frac{2x-3}{\sqrt{41}}+C.$$

उत्तर

प्रश्न 15. $\dfrac{1}{\sqrt{(x-a)(x-b)}}$.

हल :

$$\int \frac{dx}{\sqrt{(x-a)(x-b)}} = \int \frac{dx}{\sqrt{x^2-(a+b)x+ab}}$$

$$= \int \frac{dx}{\sqrt{x^2-(a+b)x+\left(\frac{a+b}{2}\right)^2-\left(\frac{a+b}{2}\right)^2+ab}}$$

$$= \int \frac{dx}{\sqrt{\left(x-\frac{a+b}{2}\right)^2-\left(\frac{a+b}{2}\right)^2+ab}}$$

$$= \int \frac{dx}{\sqrt{\left(x-\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2}}$$

$$\left[\because \int \frac{1}{\sqrt{x^2-a^2}}dx = \log\left[\left(x+\sqrt{x^2-a^2}\right)\right]\right]$$

$$= \log\left(x-\frac{a+b}{2}+\sqrt{\left(x-\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2}\right)+C$$

$$= \log\left|x-\frac{a+b}{2}+\sqrt{x^2-(a+b)x+ab}\right|+C$$

$$= \log\left|x-\frac{a+b}{2}+\sqrt{(x-a)(x-b)}\right|+C.$$ **उत्तर**

प्रश्न 16. $\dfrac{4x+1}{\sqrt{2x^2+x-3}}$.

हल :

$$\int \frac{4x+1}{\sqrt{2x^2+x-3}}dx$$

$\because$ $2x^2+x-3=t$ रखने पर,

$\therefore$ $(4x+1)\,dx = dt$

$$= \int \frac{dt}{\sqrt{t}} = \frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+C = 2\sqrt{t}+C$$

$$= 2\sqrt{2x^2+x-3}+C.$$ **उत्तर**

प्रश्न 17. $\dfrac{x+2}{\sqrt{x^2-1}}$.

हल :

$$\int \frac{x+2}{\sqrt{x^2-1}}dx = \int \frac{x}{\sqrt{x^2-1}}dx + \int \frac{2}{\sqrt{x^2-1}}dx$$

$$= I_1 + I_2$$ (मान लीजिए)

$$I_1 = \frac{1}{2}\int \frac{2x\,dx}{\sqrt{x^2-1}}$$

I_1 में $x^2 - 1 = t$ रखने पर
$2x\,dx = dt$

$$= \frac{1}{2}\int \frac{dt}{\sqrt{t}} = \frac{1}{2}\cdot\frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}$$

$$= \sqrt{t} = \sqrt{x^2-1}$$

तथा $$I_2 = 2\int \frac{1}{\sqrt{x^2-1}}dx = 2\log\left|x+\sqrt{x^2-1}\right|$$

$\therefore$ $$I_1 + I_2 = \sqrt{x^2-1} + 2\log\left|\, x+\sqrt{x^2-1}\,\right| + C.$$ **उत्तर**

प्रश्न 18. $\mathbf{\dfrac{5x-2}{1+2x+3x^2}}$.

हल : $$\int \frac{5x-2}{1+2x+3x^2}dx$$

अब $$5x - 2 = A\frac{d}{dx}(1+2x+3x^2) + B$$
$$= A(2+6x) + B = 6Ax + (2A+B)$$

x तथा अचर संख्याओं की तुलना करने पर

$$5 = 6A \text{ या } A = \frac{5}{6}$$

$$2A + B = -2 \text{ या } B = -2 - 2A = -2 - 2\times\frac{5}{6}$$

$$B = -2 - \frac{5}{3} = -\frac{11}{3}$$

$$= \int \frac{\frac{5}{6}(2+6x) - \frac{11}{3}}{1+2x+3x^2}dx$$

$$= \frac{5}{6}\int \frac{2+6x}{1+2x+3x^2}dx - \frac{11}{3}\int \frac{dx}{1+2x+3x^2}$$

$$= I_1 - \frac{11}{3}I_2 \quad \text{(मान लीजिए)} \qquad ...(i)$$

$\therefore$ $$I_1 = \frac{5}{6}\int \frac{2+6x}{1+2x+3x^2}dx$$ माना $1 + 2x + 3x^2 = t$
$\therefore (2+6x)\,dx = dt$

$$= \frac{5}{6}\int \frac{dt}{t} = \frac{5}{6}\log|t|$$

$$= \frac{5}{6}\log|1+2x+3x^2| \qquad ....(ii)$$

तथा
$$I_2 = \int \frac{dx}{1+2x+3x^2}$$
$$= \frac{1}{3}\int \frac{dx}{\frac{1}{3}+\frac{2}{3}x+x^2}$$
$$= \frac{1}{3}\int \frac{dx}{\left(x^2+\frac{2}{3}x+\frac{1}{9}\right)+\frac{1}{3}-\frac{1}{9}}$$
$$= \frac{1}{3}\int \frac{dx}{\left(x+\frac{1}{3}\right)^2+\frac{2}{9}}$$
$$= \frac{1}{3}.\frac{1}{\frac{\sqrt{2}}{3}}\tan^{-1}\left(\frac{x+\frac{1}{3}}{\frac{\sqrt{2}}{3}}\right)$$
$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) \quad ....(iii)$$

समी. (ii) और (iii) का मान समी. (i) में रखने पर
$$= \frac{5}{6}\log|1+2x+3x^2| - \frac{11}{3}\times\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)+C$$
$$= \frac{5}{6}\log|1+2x+3x^2| - \frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)+C.$$
उत्तर

प्रश्न 19. $\dfrac{6x+7}{\sqrt{(x-5)(x-4)}}$.

हल : $\displaystyle\int \frac{6x+7}{\sqrt{(x-5)(x-4)}}dx = \int \frac{6x+7}{\sqrt{x^2-9x+20}}dx$

अब
$$6x+7 = A\frac{d}{dx}(x^2-9x+20)+B = A(2x-9)+B$$

x के गुणांक तथा अचर संख्याओं की तुलना करने पर
$$6 = 2A \text{ या } A = 3$$
$$7 = -9A+B$$
$$= -27+B \quad [\because A = 3]$$
$\therefore$
$$B = 27+7 = 34$$
$$= 3\int \frac{2x-9}{\sqrt{x^2-9x+20}}dx + 34\int \frac{1}{\sqrt{x^2-9x+20}}dx$$
$$= 3I_1 + 34I_2 \text{ (मान लीजिए)} \quad ....(i)$$

अब
$$I_1 = \int \frac{2x-9}{\sqrt{x^2-9x+20}}dx$$

$\because x^2-9x+20 = t$ रखने पर,

$\therefore \quad (2x-9)\,dx = dt$

$$= \int \frac{dt}{\sqrt{t}} = \frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = 2\sqrt{t}$$

$$= 2\sqrt{x^2 - 9x + 20} \quad ....(ii)$$

तथा $$I_2 = \int \frac{1}{\sqrt{x^2 - 9x + 20}} dx$$

$$= \frac{dx}{\sqrt{x^2 - 9x + \frac{81}{4} + 20 - \frac{81}{4}}}$$

$$= \frac{dx}{\sqrt{\left(x - \frac{9}{2}\right)^2 - \frac{1}{4}}}$$

$$\left[\because \frac{1}{\sqrt{x^2 - a^2}} dx = \log\left|x + \sqrt{x^2 - a^2}\right|\right]$$

$$= \log\left| x - \frac{9}{2} + \sqrt{\left(x - \frac{9}{2}\right)^2 - \frac{1}{4}} \right|$$

$$= \log\left| x - \frac{9}{2} + \sqrt{x^2 - 9x + 20} \right| \quad ....(iii)$$

समी. (ii) व (iii) का मान समी. (i) में रखने पर

$$\therefore \quad 3I_1 + 34I_2 = 3 \times 2\sqrt{x^2 - 9x + 20} + 34\log\left| x - \frac{9}{2} + \sqrt{x^2 - 9x + 20} \right| + C$$

$$= 6\sqrt{x^2 - 9x + 20} + 34\log\left| x - \frac{9}{2} + \sqrt{x^2 - 9x + 20} \right| + C.$$ **उत्तर**

प्रश्न 20. $\frac{x+2}{\sqrt{4x - x^2}}$.

हल : $$\int \frac{x+2}{\sqrt{4x - x^2}} dx = \int \frac{x+2}{\sqrt{-(x^2 - 4x + 4) + 4}} dx$$

$$= \int \frac{x+2}{\sqrt{4 - (x-2)^2}} dx = \int \frac{x - 2 + 4}{\sqrt{4 - (x-2)^2}} dx$$

$$= \int \frac{x-2}{\sqrt{4 - (x-2)^2}} dx + 4\int \frac{1}{\sqrt{4 - (x-2)^2}} dx$$

$\because 4 - (x-2)^2 = t$ रखने पर

$\therefore -2(x-2)\, dx = dt$

$$= -\frac{1}{2}\int \frac{dt}{\sqrt{t}} + 4\sin^{-1}\left(\frac{x-2}{2}\right) \qquad \left[\because \int \frac{dt}{a^2 - x^2} = \sin^{-1}\frac{x}{a}\right]$$

$$= -\frac{1}{2}\frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + 4\sin^{-1}\left(\frac{x-2}{2}\right) + C$$

$$= -\sqrt{t} + 4\sin^{-1}\left(\frac{x-2}{2}\right) + C$$

$$= -\sqrt{4-(x-2)^2} + 4\sin^{-1}\left(\frac{x-2}{2}\right) + C$$

$$= -\sqrt{4x-x^2} + 4\sin^{-1}\left(\frac{x-2}{2}\right) + C.$$ **उत्तर**

प्रश्न 21. $\mathbf{\frac{x+2}{\sqrt{x^2+2x+3}}}$.

हल : $$\int \frac{x+2}{\sqrt{x^2+2x+3}}dx = \int \frac{\frac{1}{2}(2x+2)+1}{\sqrt{x^2+2x+3}}dx$$

$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}}dx + \int \frac{dx}{\sqrt{x^2+2x+3}}$$

अब प्रथम समाकलन में $x^2 + 2x + 3 = t$ रखने पर

$\therefore$ $(2x + 2)\, dx = dt$

$$= \frac{1}{2}\int \frac{dt}{\sqrt{t}} + \int \frac{dx}{(x+1)^2+2}$$

$$= \frac{1}{2}\frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + \log\left|(x+1) + \sqrt{(x+1)^2+2}\right| + C$$

$$= \sqrt{t} + \log\left|(x+1) + \sqrt{x^2+2x+3}\right| + C$$

$$= \sqrt{x^2+2x+3} + \log\left|(x+1) + \sqrt{x^2+2x+3}\right| + C.$$ **उत्तर**

प्रश्न 22. $\mathbf{\frac{x+3}{x^2-2x-5}}$.

हल : $\int \frac{x+3}{x^2-2x-5}dx$

अब $$x + 3 = A\frac{d}{dx}(x^2-2x-5) + B$$

$$= A(2x-2) + B$$

x तथा अचर संख्याओं की तुलना करने पर,

$$1 = 2A \text{ या } A = \frac{1}{2}$$

$$3 = -2A + B, B = 2A + 3 = 1 + 3 = 4$$

$$= \int \frac{\frac{1}{2}(2x-2)+4}{x^2-2x-5}dx$$

$$= \frac{1}{2}\int \frac{2x-2}{x^2-2x-5}dx + 4\int \frac{1}{x^2-2x-5}dx$$

पहले समाकलन में $x^2 - 2x - 5 = t$ रखने पर

$\therefore$ $(2x - 2)\,dx = dt$

$$= \frac{1}{2}\int \frac{dt}{t} + 4\int \frac{1}{(x-1)^2-6}dx$$

$$= \frac{1}{2}\log|t| + 4.\frac{1}{2\sqrt{6}}\log\frac{(x-1)-\sqrt{6}}{x-1+\sqrt{6}} + C$$

$$\left[\because \int \frac{dx}{x^2-a^a} = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)\right]$$

$$= \frac{1}{2}\log|x^2-2x-5| + \frac{2}{\sqrt{6}}\log\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}} + C. \text{ उत्तर}$$

प्रश्न 23. $\dfrac{5x+3}{\sqrt{x^2+4x+10}}$.

हल : $\displaystyle\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx$

$$5x + 3 = A\frac{d}{dx}(x^2+4x+10) + B$$

$$= A(2x + 4) + B$$

x तथा अचर संख्याओं के दोनों पक्षों की तुलना करने पर

$$5 = 2A \quad \text{ या } A = \frac{5}{2}$$

$$3 = 4A + B \quad \text{ या } B = 3 - 4A = 3 - 10 = -7$$

$$= \int \frac{\frac{5}{2}(2x+4)-7}{\sqrt{x^2+4x+10}}dx$$

$$= \frac{5}{2}\int \frac{2x+4}{\sqrt{x^2+4x+10}}dx - 7\int \frac{dx}{\sqrt{x^2+4x+10}}$$

प्रथम समाकलन में $x^2 + 4x + 10 = t$ रखने पर तथा $(2x + 4)\,dx = dt$

$$I = \frac{5}{2}\int \frac{dt}{\sqrt{t}} - 7\int \frac{dx}{\sqrt{(x+2)^2+6}}$$

$$= \frac{5}{2} \cdot \frac{t^{-\frac{1}{2}+1}}{\sqrt{t}} - 7 \log \left| (x+2) + \sqrt{(x+2)^2 + 6} \right| + C$$

$$= 5\sqrt{t} - 7 \log \left| (x+2) + \sqrt{x^2 + 4x + 10} \right| + C$$

$$= 5\sqrt{x^2 + 4x + 10} - 7 \log \left| (x+2) + \sqrt{x^2 + 4x + 10} \right| + C.$$

उत्तर

प्रश्न 24 व 25 में सही उत्तर का चयन कीजिए—

प्रश्न 24. $\int \frac{dx}{x^2 + 2x + 2}$ **बराबर है—**

(A) $x \tan^{-1} (x+1) + C$
(B) $\tan^{-1} (x+1) + C$
(C) $(x+1) \tan^{-1} x + C$
(D) $\tan^{-1} x + C$

हल :

$$\int \frac{dx}{x^2 + 2x + 2} = \int \frac{dx}{(x+1)^2 + 1}$$

$$= \int \frac{dx}{(x+1)^2 + (1)^2}$$

$$= \tan^{-1} (x+1) + C$$

अत: विकल्प (B) सही है।

उत्तर

प्रश्न 25. $\int \frac{dx}{\sqrt{9x - 4x^2}}$ **बराबर है—**

(A) $\frac{1}{9} \sin^{-1} \left(\frac{9x - 8}{8} \right) + C$
(B) $\frac{1}{2} \sin^{-1} \left(\frac{8x - 9}{9} \right) + C$
(C) $\frac{1}{3} \sin^{-1} \left(\frac{9x - 8}{8} \right) + C$
(D) $\frac{1}{2} \sin^{-1} \left(\frac{9x - 8}{8} \right) + C.$

हल :

$$\int \frac{dx}{\sqrt{9x - 4x^2}} = \int \frac{dx}{\sqrt{-(4x^2 - 9x)}}$$

$$= \int \frac{dx}{\sqrt{-\left(4x^2 - 9x + \frac{81}{16}\right) - \frac{81}{16}}}$$

$$= \int \frac{dx}{\sqrt{\frac{81}{16} - \left(2x - \frac{9}{4}\right)^2}}$$

मान लीजिए $2x - \frac{9}{4} = t$

तब $2dx = dt$

$$= \frac{1}{2} \int \frac{dt}{\left(\frac{9}{4}\right)^2 - (t)^2} = \frac{1}{2} \sin^{-1} \frac{t}{9/4} + C$$

$$= \frac{1}{2}\sin^{-1}\frac{4t}{9} + C$$

$$= \frac{1}{2}\sin^{-1}\frac{4\left(2x - \frac{9}{4}\right)}{9} + C$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{8x - 9}{9}\right) + C$$

अत: विकल्प (B) सही है। **उत्तर**

प्रश्नावली 7·5

प्रश्न 1 से 21 तक के प्रश्नों में परिमेय फलनों का समाकलन कीजिए—

प्रश्न 1. $\frac{x}{(x+1)(x+2)}$.

हल : मान लीजिए $\frac{x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$

$\therefore$ $x = A(x+2) + B(x+1)$

$x = -1$ रखने पर, $-1 = A.1$ अर्थात् $A = -1$

$x = -2$ रखने पर, $-2 = B(-2+1)$ अर्थात् $B = 2$

$\therefore$ $\frac{x}{(x+1)(x+2)} = -\frac{1}{x+1} + \frac{2}{(x+2)}$

$$\int \frac{x}{(x+1)(x+2)}dx = -\int \frac{1}{x+1}dx + 2\int \frac{1}{x+2}dx$$

$$= -\log|x+1| + 2\log|x+2| + C$$

$$= -\log|x+1| + \log(x+2)^2$$

$$= \log\frac{(x+2)^2}{|x+1|} + C.$$

उत्तर

प्रश्न 2. $\frac{1}{x^2 - 9}$.

हल : $\frac{1}{x^2-9} = \frac{1}{(x-3)(x+3)} = \frac{A}{x-3} + \frac{B}{x+3}$

$$1 = A(x+3) + B(x-3)$$

$x = 3$ रखने पर, $1 = A.6$ या $A = \frac{1}{6}$

$x = -3$ रखने पर, $1 = B(-6)$ या $B = -\frac{1}{6}$

$\therefore$ $\frac{1}{x^2-9} = \frac{1}{6(x-3)} - \frac{1}{6(x+3)}$

$\therefore$ $\int \frac{dx}{x^2-9} = \frac{1}{6}\int \frac{dx}{x-3}dx - \frac{1}{6}\int \frac{1}{x+3}dx$

$$= \frac{1}{6}\log|x-3| - \frac{1}{6}\log|x+3| + C$$

$$= \frac{1}{6}\log\left|\frac{x-3}{x+3}\right| + C.$$ उत्तर

प्रश्न 3. $\dfrac{3x-1}{(x-1)(x-2)(x-3)}$.

हल : $$\frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

$$3x-1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$$

$x = 1$ रखने पर, $2 = A(-1)(-2)$ या $A = 1$

$x = 2$ रखने पर, $5 = B.1(-1)$ या $B = -5$

$x = 3$ रखने पर, $8 = C.2.1$ या $C = 4$

अब $$\frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{1}{x-1} - \frac{5}{x-2} + \frac{4}{x-3}$$

$$\int\frac{3x-1}{(x-1)(x-2)(x-3)}dx = \int\frac{1}{x-1}dx - 5\int\frac{1}{x-2}dx + 4\int\frac{1}{x-3}dx$$

$$= \log|x-1| - 5\log|x-2| + 4\log|x-3| + C.$$ उत्तर

प्रश्न 4. $\dfrac{x}{(x-1)(x-2)(x-3)}$.

हल : मान लीजिए $$\frac{x}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

$$x = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$$

$x = 1$ लेने पर $1 = A(-1)(-2)$ या $A = \frac{1}{2}$

$x = 2$ लेने पर $2 = B.1(-1)$ या $B = -2$

$x = 3$ लेने पर $3 = C.2.1$ या $2C = 3$ या $C = \frac{3}{2}$

$\therefore$ $$\frac{x}{(x-1)(x-2)(x-3)} = \frac{1}{2(x-1)} - \frac{2}{(x-2)} + \frac{3}{2(x-3)}$$

$$\int\frac{x}{(x-1)(x-2)(x-3)}dx = \frac{1}{2}\int\frac{1}{x-1}dx - 2\int\frac{1}{x-2}dx + \frac{3}{2}\int\frac{1}{x-3}dx$$

$$= \frac{1}{2}\log|x-1| - 2\log|x-2|$$

$$+ \frac{3}{2}\log|x-3| + C.$$ उत्तर

प्रश्न 5. $\dfrac{2x}{x^2+3x+2}$.

हल : मान लीजिए $$\frac{2x}{x^2+3x+2} = \frac{2x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

$\therefore$ $$2x = A(x+2) + B(x+1)$$

$x = -1$ लेने पर $\qquad -2 = A(1)$ अर्थात् $A = -2$

$x = -2$ लेने पर $\qquad -4 = B(-1)$ अर्थात् $B = 4$

$$\therefore \quad \frac{2x}{(x+1)(x+2)} = \frac{-2}{x+1} + \frac{4}{x+2}$$

$$\int \frac{2x}{(x+1)(x+2)} dx = -2\int \frac{1}{x+1} dx + 4\int \frac{1}{x+2} dx$$

$$= -2 \log|x+1| + 4 \log|x+2| + C$$

$$= 4 \log|x+2| - 2 \log|x+1| + C.$$

उत्तर

प्रश्न 6. $\frac{1-x^2}{x(1-2x)}$.

हल : $\quad \frac{1-x^2}{x(1-2)} = \frac{1-x^2}{-2x^2+x}$

अब अंश को हर से भाग देने पर,

अर्थात् $\quad \frac{1-x^2}{x(1-2x)} = \frac{1}{2} + \frac{1-\frac{1}{2}x}{x-2x^2} = \frac{1}{2} + \frac{2-x}{2x(1-2x)} \quad$ ...(i)

मान लीजिए $\quad \frac{2-x}{x(1-2x)} = \left[\frac{A}{x} + \frac{B}{1-2x}\right]$

$\therefore \quad 2 - x = A(1-2x) + Bx$

$x = 0$ लेने पर, $\quad 2 = A$ अर्थात् $A = 2$

$x = \frac{1}{2}$ लेने पर, $\quad 2 - \frac{1}{2} = 0 + B \times \frac{1}{2}$

या $\quad \frac{B}{2} = \frac{3}{2}$

या $\quad B = 3$

अत: $\quad \frac{2-x}{x(1-2x)} = \frac{2}{x} + \frac{3}{1-2x}$.

समीकरण (i) से,

$$\frac{1-x^2}{x(1-2x)} = \frac{1}{2} + \frac{2-x}{2x(1-2x)}$$

$$= \frac{1}{2} + \frac{1}{2}\left[\frac{2}{x} + \frac{3}{1-2x}\right]$$

$$= \frac{1}{2} + \frac{1}{x} + \frac{3}{2(1-2x)}$$

$$\therefore \quad \int \frac{1-x^2}{x(1-2x)} dx = \frac{1}{2}\int dx + \int \frac{1}{x} dx + \frac{3}{2}\int \frac{1}{1-2x} dx$$

$$= \frac{1}{2}x + \log|x| + \frac{3}{2}\left(\frac{1}{-2}\right)\log|1-2x| + C$$

$$= \frac{1}{2}x + \log|x| - \frac{3}{4}\log|1-2x| + C.$$

उत्तर

प्रश्न 7. $\dfrac{x}{(x^2+1)(x-1)}$.

हल : मान लीजिए $\quad \dfrac{x}{(x^2+1)(x-1)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+1}$

$\therefore \quad x = A(x^2+1) + (Bx+C)(x-1)$

$x = A(x^2+1) + B(x^2-x) + C(x-1)$

$x = 1$ रखने पर, $\quad 1 = A \times 2$ या $A = \dfrac{1}{2}$

x^2 तथा x के गुणांक की तुलना करने पर,

$$0 = A + B \text{ या } B = -A = -\frac{1}{2}$$

तथा $\quad 1 = -B + C$ या $C = 1 + B = 1 - \dfrac{1}{2} = \dfrac{1}{2}$

अत: $\quad \dfrac{x}{(x^2+1)(x-1)} = \dfrac{1}{2(x-1)} + \dfrac{-\frac{1}{2}x + \frac{1}{2}}{x^2+1}$

$$= \frac{1}{2(x-1)} - \frac{1}{2}.\frac{x-1}{x^2+1}$$

$$= \frac{1}{2(x-1)} - \frac{x}{2(x^2+1)} + \frac{1}{2(x^2+1)}$$

$$\int \frac{x\,dx}{(x^2+1)(x-1)} = \frac{1}{2}\int \frac{1}{x-1}dx - \frac{1}{4}\int \frac{2x}{x^2+1}dx + \frac{1}{2}\int \frac{1}{x^2+1}dx$$

$$= \frac{1}{2}\log|x-1| - \frac{1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1}x + C.$$ **उत्तर**

प्रश्न 8. $\dfrac{x}{(x-1)^2(x+2)}$.

हल : मान लीजिए

$$\frac{x}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

$\therefore \quad x = A(x-1)(x+2) + B(x+2) + C(x-1)^2$

$x = 1$ रखने पर, $\quad 1 = B \times 3$ या $B = \dfrac{1}{3}$

$x = -2$ रखने पर, $\quad -2 = C(-3)^2$ या $C = -\dfrac{2}{9}$

x^2 के गुणांक की तुलना करने पर,

$$0 = A + C \text{ या } A = -C = \frac{2}{9}$$

$\therefore \quad \dfrac{x}{(x-1)^2(x+2)} = \dfrac{2}{9(x-1)} + \dfrac{1}{3(x-1)^2} - \dfrac{2}{9(x+2)}$

या $$\int \frac{x}{(x-1)^2(x+2)}dx = \frac{2}{9}\int\frac{dx}{x-1} + \frac{1}{3}\int\frac{1}{(x-1)^2}dx - \frac{2}{9}\int\frac{1}{x+2}dx$$

$$= \frac{2}{9}\log|x-1| + \frac{1}{3}\frac{(x-1)^{-2+1}}{-2+1} - \frac{2}{9}\log|x+2| + C$$

$$= \frac{2}{9}\log|x-1| - \frac{1}{3(x-1)} - \frac{2}{9}\log|x+2| + C$$

$$= \frac{2}{9}\log\left|\frac{x-1}{x+2}\right| - \frac{1}{3(x-1)} + C.$$ उत्तर

प्रश्न 9. $\dfrac{3x+5}{x^3-x^2-x+1}$.

हल : ∵ $$\frac{3x+5}{x^3-x^2-x+1} = \frac{3x+5}{x^2(x-1)-(x-1)} = \frac{3x+5}{(x-1)(x^2-1)}$$

$$= \frac{3x+5}{(x-1)(x-1)(x+1)}$$

∴ $$\frac{3x+5}{(x-1)^2(x+1)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

∴ $$3x+5 = A(x-1)^2 + B(x+1)(x-1) + C(x+1)$$

$x = 1$ लेने पर, $8 = C \times 2$ या $C = 4$

$x = -1$ लेने पर, $2 = A \times 4$ या $A = \frac{1}{2}$

अब x^2 के गुणांक की तुलना करने पर

$$0 = A + B \text{ या } B = -A = -\frac{1}{2}$$

अत: $$\frac{3x+5}{x^3-x^2-x+1} = \frac{1}{2(x+1)} - \frac{1}{2(x-1)} + \frac{4}{(x-1)^2}$$

या $$\int\frac{3x+5}{x^3-x^2-x+1}dx = \frac{1}{2}\int\frac{1}{x+1}dx - \frac{1}{2}\int\frac{1}{x-1}dx + 4\int\frac{1}{(x-1)^2}dx$$

$$= \frac{1}{2}\log|x+1| - \frac{1}{2}\log|x-1| + 4\frac{(x-1)^{-2+1}}{-2+1} + C$$

$$= \frac{1}{2}\log\left|\frac{x+1}{x-1}\right| - \frac{4}{x-1} + C.$$ उत्तर

प्रश्न 10. $\dfrac{2x-3}{(x^2-1)(2x+3)}$.

हल : $$\frac{2x-3}{(x^2-1)(2x+3)} = \frac{2x-3}{(x-1)(x+1)(2x+3)}$$

मान लीजिए $$\frac{2x-3}{(x^2-1)(2x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{2x+3}$$

$\therefore \quad 2x - 3 = A(x+1)(2x+3) + B(x-1)(2x+3) + C(x^2-1)$

$x = 1$ लेने पर, $\quad -1 = A \times 2 \times 5$ या $A = -\frac{1}{10}$

$x = -1$ लेने पर, $\quad -5 = B(-2)(1)$ या $B = \frac{5}{2}$

$x = -\frac{3}{2}$ लेने पर, $\quad -6 = C\left(\frac{9}{4} - 1\right) = C\left(\frac{5}{4}\right)$

$\therefore \quad C = \frac{-6 \times 4}{5} = -\frac{24}{5}$

अत: $\quad \frac{2x-3}{(x^2-1)(2x+3)} = -\frac{1}{10(x-1)} + \frac{5}{2(x+1)} - \frac{24}{5} \cdot \frac{1}{2x+3}$

या $\quad \int \frac{2x-3}{(x^2-1)(2x+3)} dx = -\frac{1}{10}\int \frac{1}{x-1} dx + \frac{5}{2}\int \frac{1}{x+1} dx - \frac{24}{5}\int \frac{1}{2x+3} dx$

$$= -\frac{1}{10}\log|x-1| + \frac{5}{2}\log|x+1| - \frac{24}{5}\frac{1}{2}\log|2x+3| + C$$

$$= \frac{5}{2}\log|x+1| - \frac{1}{10}\log|x-1| - \frac{12}{5}\log|2x+3| + C.$$ **उत्तर**

प्रश्न 11. $\frac{5x}{(x+1)(x^2-4)}$.

हल : मान लीजिए $\quad \frac{5x}{(x+1)(x^2-4)} = \frac{5x}{(x+1)(x-2)(x+2)}$

$$= \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{x+2}$$

$\therefore \quad 5x = A(x-2)(x+2) + B(x+1)(x+2) + C(x+1)(x-2)$

$x = -1$ रखने पर, $\quad -5 = A(1)(-3)$ या $A = \frac{5}{3}$

$x = -2$ रखने पर, $\quad -10 = C(-1)(-2-2)$ या $C = -\frac{10}{4} = -\frac{5}{2}$

$x = 2$ रखने पर, $\quad 10 = B(3)(4)$ या $12B = 10$ या $B = \frac{10}{12} = \frac{5}{6}$

$\therefore \quad \frac{5x}{(x+1)(x^2-4)} = \frac{5}{3(x+1)} + \frac{5}{6(x-2)} - \frac{5}{2(x+2)}$

या $\quad \int \frac{5x}{(x+1)(x^2-4)} dx = \frac{5}{3}\int \frac{1}{x+1} dx + \frac{5}{6}\int \frac{1}{x-2} dx - \frac{5}{2}\int \frac{1}{x+2} dx$

$$= \frac{5}{3}\log|x+1| + \frac{5}{6}\log|x-2| - \frac{5}{2}\log|x+2| + C.$$ **उत्तर**

प्रश्न 12. $\frac{x^3+x+1}{x^2-1}$.

हल : $\because \dfrac{x^3+x+1}{x^2-1}$ में अंश की घात हर की घात से अधिक है इसलिए

x^3+x+1 को x^2-1 से भाग देने पर

$$\therefore \quad \frac{x^3+x+1}{x^2-1} = x+\frac{2x+1}{x^2-1} = x+\frac{2x}{x^2-1}+\frac{1}{x^2-1}$$

या
$$\int\frac{x^3+x+1}{x^2-1}dx = \int x\,dx+\int\frac{2x}{x^2-1}dx+\int\frac{1}{x^2-1}dx$$

$$= \frac{x^2}{2}+\log|x^2-1|+\frac{1}{2}\log\left|\frac{x-1}{x+1}\right|+C$$

$$= \frac{x^2}{2}+\log|(x-1)(x+1)|+\frac{1}{2}\log|x-1|-\frac{1}{2}\log|x+1|+C$$

$$= \frac{x^2}{2}+\log|x-1|+\log|x+1|$$
$$+\frac{1}{2}\log|x-1|-\frac{1}{2}\log|x+1|+C$$

$$= \frac{x^2}{2}+\frac{3}{2}\log|x-1|+\frac{1}{2}\log|x+1|+C.$$ **उत्तर**

प्रश्न 13. $\dfrac{2}{(1-x)(1+x^2)}$.

हल : मान लीजिए
$$\frac{2}{(1-x)(1+x^2)} = \frac{A}{1-x}+\frac{Bx+C}{1+x^2}$$

$$2 = A(1+x^2)+(Bx+C)(1-x)$$
$$= A(1+x^2)+B(x-x^2)+C(1-x)$$

$x=1$ लेने पर, $2 = A(2)$ या $2A=2$ या $A=1$

x^2 तथा x के गुणांकों की तुलना करने पर

$0 = A-B$ या $B=A=1$

$0 = B-C$ या $C=B=1$

$$\therefore \quad \frac{2}{(1-x)(1+x^2)} = \frac{1}{1-x}+\frac{x+1}{1+x^2}$$

$$= \frac{1}{1-x}+\frac{x}{1+x^2}+\frac{1}{1+x^2}$$

$$\int\frac{2}{(1-x)(1+x^2)}dx = \int\frac{1}{1-x}dx+\frac{1}{2}\int\frac{2x}{1+x^2}dx+\int\frac{1}{1+x^2}dx$$

$$= -\int\frac{dx}{x-1}+\frac{1}{2}\int\frac{2x}{x^2+1}dx+\int\frac{2x}{x^2+1}$$

$$= -\log|x-1|+\frac{1}{2}\log(1+x^2)+\tan^{-1}x+C.$$ **उत्तर**

प्रश्न 14. $\dfrac{3x-1}{(x-2)^2}$.

हल : मान लीजिए
$$\frac{3x-1}{(x-2)^2} = \frac{A}{x-2}+\frac{B}{(x-2)^2}$$
$$3x-1 = A(x-2)+B$$
$x = 2$ लेने पर, $6-1 = B$ या $B = 5$

दोनों पक्षों के x के गुणांकों की तुलना करने पर
$$3 = A \text{ या } A = 3$$
$\therefore$
$$\frac{3x-1}{(x-2)^2} = \frac{3}{x-2}+\frac{5}{(x-2)^2}$$
या
$$\int\frac{3x-1}{(x-2)^2}dx = 3\int\frac{dx}{x-2}+5\int\frac{dx}{(x-2)^2}$$
$$= 3\log|x-2|-\frac{5}{x-2}+C.$$
उत्तर

प्रश्न 15. $\dfrac{1}{x^4-1}$.

हल : मान लीजिए
$$\frac{1}{x^4-1} = \frac{1}{(x-1)(x+1)(x^2+1)}$$
$$= \frac{A}{x-1}+\frac{B}{x+1}+\frac{Cx+D}{x^2+1}$$
$$1 = A(x+1)(x^2+1)+B(x-1)(x^2+1)+(Cx+D)(x^2-1)$$
$$1 = A(x+1)(x^2+1)+B(x-1)(x^2+1) + C(x^3-x)+D(x^2-1)$$
$x = 1$ लेने पर, $1 = A\times 2\times 2$ या $A = \dfrac{1}{4}$

$x = -1$ रखने पर, $1 = B(-2)(2)$ या $-4B = 1$ या $B = -\dfrac{1}{4}$

x^3 तथा x^2 के गुणांकों की तुलना करने पर,
$$0 = A+B+C \text{ या } C = -A-B = -\frac{1}{4}+\frac{1}{4} = 0$$
$$0 = A-B+D \text{ या } D = B-A = -\frac{1}{4}-\frac{1}{4} = \frac{-2}{4} = -\frac{1}{2}$$
$\therefore$
$$\frac{1}{x^4-1} = \frac{1}{4(x-1)}-\frac{1}{4(x+1)}-\frac{1}{2(x^2+1)}$$
या
$$\int\frac{1}{x^4-1}dx = \frac{1}{4}\int\frac{1}{x-1}dx-\frac{1}{4}\int\frac{1}{x+1}dx-\frac{1}{2}\int\frac{1}{x^2+1}dx$$
$$= \frac{1}{4}\log|x-1|-\frac{1}{4}\log|x+1|-\frac{1}{2}\tan^{-1}x+C$$
$$= \frac{1}{4}\log\left|\frac{x-1}{x+1}\right|-\frac{1}{2}\tan^{-1}x+C.$$
उत्तर

प्रश्न 16. $\dfrac{1}{x(x^n+1)}$.

हल : मान लीजिए $\displaystyle\int\frac{1}{x(x^n+1)}dx = \int\frac{x^{n-1}}{x^{n-1}.x(x^n+1)}dx = \int\frac{x^{n-1}dx}{x^n(x^n+1)}$

यहाँ $x^n = t$ रखने पर $\quad n\,x^{n-1}\,dx = dt$

$$= \frac{1}{n}\int\frac{dt}{t(t+1)}$$

अब $\quad \dfrac{1}{t(t+1)} = \dfrac{A}{t}+\dfrac{B}{t+1}$

$\therefore \quad 1 = A(t+1)+Bt$

$t = 0$ लेने पर, $\quad 1 = A$ या $A = 1$

$t = -1$ लेने पर, $\quad 1 = B(-1)$ या $B = -1$

$\therefore \quad \dfrac{1}{t(t+1)} = \dfrac{1}{t}-\dfrac{1}{t+1}$

$$\frac{1}{n}\int\frac{dt}{t(t+1)} = \frac{1}{n}\int\left(\frac{1}{t}-\frac{1}{t+1}\right)dt$$

$$= \frac{1}{n}[\log|t|-\log|t+1|]+C$$

$$= \frac{1}{n}\log\left|\frac{t}{t+1}\right|+C$$

$$= \frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right|+C.$$

उत्तर

प्रश्न 17. $\dfrac{\cos x}{(1-\sin x)(2-\sin x)}$.

हल : $\displaystyle\int\frac{\cos x}{(1-\sin x)(2-\sin x)}dx$

$\because$ $\sin x = t$ रखने पर

$\therefore \quad \cos x\,dx = dt$

$$= \int\frac{dt}{(1-t)(2-t)}$$

अब $\quad \dfrac{1}{(1-t)(2-t)} = \dfrac{A}{1-t}+\dfrac{B}{2-t} \quad$ ....(i)

$\therefore \quad 1 = A(2-t)+B(1-t)$

$t = 1$ लेने पर, $\quad 1 = A\times 1$ या $A = 1$

$t = 2$ लेने पर, $\quad 1 = B(1-2)$ या $-B = 1$ या $B = -1$

A तथा B के मान समीकरण (i) में रखने पर

$\therefore \quad \dfrac{1}{(1-t)(2-t)} = \dfrac{1}{1-t}-\dfrac{1}{2-t}$

$$\int \frac{dt}{(1-t)(2-t)}$$

$$= \int \frac{1}{1-t} dt - \int \frac{1}{2-t} dt$$

$$= -\log|1-t| + \log|2-t| + C$$

$$= \log\left|\frac{2-t}{1-t}\right| + C$$

$$= \log\left|\frac{2-\sin x}{1-\sin x}\right| + C.$$ उत्तर

प्रश्न 18. $\dfrac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$.

हल : $\dfrac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$ में $x^2 = t$ रखने पर

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = \frac{(t+1)(t+2)}{(t+3)(t+4)} = \frac{t^2+3t+2}{t^2+7t+12}$$

इस परिमेय में अंश व हर की घात बराबर है इसलिए अंश को हर से भाग देने पर

$$\therefore \quad \frac{t^2+3t+2}{t^2+7t+12} = 1 - \frac{4t+10}{t^2+7t+12}$$

$$= 1 - \frac{2(2t+5)}{t^2+7t+12} \quad \text{....(i)}$$

अब लीजिए $\quad \dfrac{2t+5}{t^2+7t+12} = \dfrac{2t+5}{(t+3)(t+4)} = \dfrac{A}{t+3} + \dfrac{B}{t+4}$

$\therefore \quad 2t + 5 = A(t+4) + B(t+3)$

$t = -3$ लेने पर, $\quad -1 = A \times 1$ या $A = -1$

$t = -4$ लेने पर, $\quad -3 = B(-4+3)$ या $B = 3$

$$\therefore \quad \frac{2t+5}{t^2+7t+12} = -\frac{1}{t+3} + \frac{3}{t+4}$$

समीकरण (i) में $t = x^2$ रखने पर,

$$\int \left\{1 - 2\left(-\frac{1}{x^2+3} + \frac{3}{x^2+4}\right)\right\} dx$$

$$= \int dx + 2\int \frac{1}{x^2+3} dx - 6\int \frac{1}{x^2+4} dx$$

$$= \int dx + 2\int \frac{1}{x^2+(\sqrt{3})^2} dx - 6\int \frac{1}{x^2+(2)^2} dx$$

$$\left[\because \int \frac{1}{x^2+a^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

$$= x + 2.\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} - \frac{6}{2}\tan^{-1}\frac{x}{2} + C$$

$$= x + \frac{2}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} - 3\tan^{-1}\frac{x}{2} + C.$$ **उत्तर**

प्रश्न 19. $\frac{2x}{(x^2+1)(x^2+3)}$.

हल : $\int \frac{2x}{(x^2+1)(x^2+3)}dx$, मान लीजिए $x^2 = t$, $\therefore$ $2x\ dt = dt$

$$= \int \frac{dt}{(t+1)(t+3)}$$

अब $\frac{1}{(t+1)(t+3)} = \frac{A}{t+1} + \frac{B}{t+3}$

$$1 = A(t+3) + B(t+1)$$

$t = -1$ लेने पर, $1 = A \times 2$ या $A = \frac{1}{2}$

$t = -3$ लेने पर, $1 = B(-3+1)$ या $-2B = 1$ या $B = -\frac{1}{2}$

$\therefore$ $\frac{1}{(t+1)(t+3)} = \frac{1}{2(t+1)} - \frac{1}{2(t+3)}$

$\therefore$ $\int \frac{1}{(t+1)(t+3)}dt = \frac{1}{2}\int \frac{1}{t+1}dt - \frac{1}{2}\int \frac{1}{t+3}dt$

$$= \frac{1}{2}\log|t+1| - \frac{1}{2}\log|t+3| + C$$

$$= \frac{1}{2}\log\left|\frac{t+1}{t+3}\right| + C$$

$$= \frac{1}{2}\log\left|\frac{x^2+1}{x^2+3}\right| + C.$$ **उत्तर**

प्रश्न 20. $\frac{1}{x(x^4-1)}$.

हल : $\int \frac{1}{x(x^4-1)}dx = \int \frac{x^3}{x^4(x^4-1)}dx$

$\because$ $x^4 = t$ रखने पर

$\therefore$ $4x^3\ dx = dt$

$$= \frac{1}{4}\int \frac{dt}{t(t-1)} \quad ....(i)$$

अब लीजिए $\frac{1}{(t-1)t} = \frac{A}{t-1} + \frac{B}{t}$

$$1 = At + B(t-1)$$

$t = 0$ लेने पर, $\quad 1 = B(-1)$ या $B = -1$

$t = 1$ लेने पर, $\quad 1 = A \times 1$ या $A = 1$

$$\therefore \quad \frac{1}{(t-1)t} = \frac{1}{t-1} - \frac{1}{t}$$

अब समीकरण (i) से,

$$= \frac{1}{4}\int\left(\frac{1}{t-1} - \frac{1}{t}\right)dt$$

$$= \frac{1}{4}\int\frac{1}{t-1}dt - \frac{1}{4}\int\frac{1}{t}dt$$

$$= \frac{1}{4}\log|t-1| - \frac{1}{4}\log|t| + C$$

$$= \frac{1}{4}\log\left|\frac{t-1}{t}\right| + C$$

$$= \frac{1}{4}\log\left|\frac{x^4-1}{x^4}\right| + C.$$

उत्तर

प्रश्न 21. $\dfrac{\mathbf{1}}{\mathbf{(e^x - 1)}}$.

हल : $$\int\frac{dx}{(e^x-1)} = \int\frac{e^x dx}{e^x(e^x-1)}$$

मान लीजिए $e^x = t$ हो, तब $e^x\,dx = dt$

$$= \int\frac{dt}{t(t-1)}$$

अब लीजिए $$\frac{1}{t(t-1)} = \frac{A}{t-1} + \frac{B}{t}$$

$$1 = At + B(t-1)$$

$t = 1$ लेने पर, $\quad 1 = A \times 1$ या $A = 1$

$t = 0$ लेने पर, $\quad 1 = B(-1)$ या $B = -1$

$$\therefore \quad \frac{1}{t(t-1)} = \frac{1}{t-1} - \frac{1}{t}$$

या $$\int\frac{1}{t(t-1)}dt$$

$$= \int\frac{1}{t-1}dt - \int\frac{1}{t}dt$$

$$= \log|t-1| - \log|t| + C$$

$$= \log\frac{|t-1|}{|t|} + C$$

$$= \log\left|\frac{e^x-1}{e^x}\right| + C.$$

($t = e^x$ रखने पर) उत्तर

प्रश्न 22 व 23 में सही उत्तर का चयन कीजिए—

प्रश्न 22. $\int \frac{x\,dx}{(x-1)(x-2)}$ **बराबर है—**

(A) $\log\left|\frac{(x-1)^2}{x-2}\right| + C$ (B) $\log\left|\frac{(x-2)^2}{x-1}\right| + C$

(C) $\log\left|\left(\frac{x-1}{x-2}\right)^2\right| + C$ (D) $\log|(x-1)(x-2)| + C$

हल : $\frac{x}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$

या $x = A(x-2) + B(x-1)$

$x = 1$ लेने पर, $1 = A(1-2)$

या $-A = 1$ या $A = -1$

$x =$ लेने पर, $2 = B(2-1)$

या $B = 2$

अब $\frac{x}{(x-1)(x-2)} = -\frac{1}{x-1} + \frac{2}{x-2}$

$\therefore$ $\int \frac{x\,dx}{(x-1)(x-2)} = -\int \frac{1}{x-1}dx + 2\int \frac{1}{x-2}dx$

$= -\log|x-1| + 2\log|x-2| + C$

$= \log\left|\frac{(x-2)^2}{x-1}\right| + C.$

अत: विकल्प (B) सही है। **उत्तर**

प्रश्न 23. $\int \frac{dx}{x(x^2+1)}$ **बराबर है—**

(A) $\log|x| - \frac{1}{2}\log(x^2+1) + C$

(B) $\log|x| + \frac{1}{2}\log(x^2+1) + C$

(C) $-\log|x| + \frac{1}{2}\log(x^2+1) + C$

(D) $\frac{1}{2}\log|x| + \log(x^2+1) + C$.

हल : $\int \frac{dx}{x(x^2+1)} = \int \frac{x\,dx}{x^2(x^2+1)}$

मान लीजिए $x^2 = t$ हो, तब $2x\,dx = dt$

$= \frac{1}{2}\int \frac{dt}{t(t+1)}$

अब $$\frac{1}{t(t+1)} = \frac{A}{t}+\frac{B}{t+1}$$

$$1 = A(t + 1) + Bt$$

$t = 0$ लेने पर,

$$1 = A \text{ या } A = 1$$

$t = -1$ लेने पर,

$$1 = B \times (-1) \text{ या } B = -1$$

$\therefore$ $$\frac{1}{t(t+1)} = \frac{1}{t} - \frac{1}{t+1}$$

अतः $$\frac{1}{2}\int\frac{dt}{t(t+1)} = \frac{1}{2}\left[\int\frac{1}{t}dt-\int\frac{1}{t+1}dt\right]$$

$$= \frac{1}{2}\log|t|-\frac{1}{2}\log|t+1|+C$$

$$= \frac{1}{2}\log|x^2|-\frac{1}{2}\log|x^2+1|+C \qquad (t = x^2 \text{ रखने पर})$$

$$= \log|x|-\frac{1}{2}\log|x^2+1|+C.$$

अतः विकल्प (A) सही है। **उत्तर**

प्रश्नावली 7·6

प्रश्न 1 से 22 तक के प्रश्नों के फलनों का समाकलन कीजिए—

प्रश्न 1. $x \sin x$.

हल : $\int x\sin x\, dx$

$$\int uv\, dx = u\int v\, dx-\int\left(\frac{du}{dx}\int v\, dx\right)dx$$

$u = x$ तथा $v = \sin x$ लेने पर,

$$= x\int \sin x\, dx-\int\left[\left(\frac{d}{dx}x\right)\int \sin x\, dx\right]dx$$

$$= x(-\cos x)-\int 1(-\cos x)\, dx$$

$$= -\cos x + \sin x + C.$$ **उत्तर**

प्रश्न 2. $x \sin 3x$.

हल : $\int x\sin 3x\, dx$

खण्डशः समाकलन से,

$$\int uv\, dx = u\int v\, dx-\int\left(\frac{du}{dx}\int v\, dx\right)dx$$

$u = x$ तथा $v = 3x$ लेने पर

$$\int x.\sin 3x\, dx = x\left(-\frac{\cos 3x}{3}\right) - \int 1.\left(-\frac{\cos 3x}{3}\right)dx$$

$$= -\frac{x\cos 3x}{3} + \frac{1}{3}\int \cos 3x\, dx$$

$$= -\frac{x\cos 3x}{3} + \frac{1}{9}\sin 3x + C$$

$$= -\frac{x}{3}\cos 3x + \frac{1}{9}\sin 3x + C.$$ उत्तर

i žu 3. x^2e^x.

हल : $\int x^2e^x dx$

खण्डशः समाकलन से,

$$\int uv\, dx = u\int v\, dx - \int\left(\frac{du}{dx}\int v\, dx\right)dx$$

$u = x^2$ तथा $v = e^x$ लेने पर

$$= x^2\int e^x dx - \int\left(\frac{d}{dx}x^2\int e^x dx\right)$$

$$= x^2e^x - \int 2xe^x dx = x^2e^x - 2\int xe^x dx$$

पुनः खण्डशः समाकलन करने पर

$$= x^2e^x - 2\left[x.\int e^x dx - \int\left(1.\int e^x dx\right)dx\right]$$

$$= x^2e^x - 2\left[xe^x - \int e^x dx\right]$$

$$= x^2e^x - 2xe^x + 2e^x + C$$

$$= e^x(x^2 - 2x + 2) + C.$$ उत्तर

प्रश्न 4. $x \log x$.

हल : $\int x\log x\, dx$

खण्डशः समाकलन से,

$$\int uv\, dx = u\int v\, dx - \int\left(\frac{du}{dx}\int v\, dx\right)dx$$

$u = \log x$ तथा $v = x$ रखने पर,

$$\int x\log x\, dx$$

$$= (\log x).\frac{x^2}{2} - \int\frac{1}{x}.\frac{x^2}{2}dx$$

$$= \frac{x^2}{2}\log x - \frac{1}{2}\int x\, dx$$

$$= \frac{x^2}{2}\log x - \frac{1}{2}.\frac{x^2}{2} + C$$

$$= \frac{x^2}{2}\log x - \frac{1}{4}x^2 + C.$$ उत्तर

प्रश्न 5. $x \log 2x$.

हल : $\int x \log 2x\, dx$

खण्डशः समाकलन से,

$$\int uv\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\int v\,dx\right)dx$$

यहाँ $u = \log 2x$ तथा $v = x$ लेने पर,

$$= \log 2x.\int x\,dx - \int\left[\frac{d}{dx}(\log 2x)\int x\,dx\right]dx$$

$$= (\log 2x) \times \frac{x^2}{2} - \int \frac{1}{x}.\frac{x^2}{2}\,dx$$

$$= \frac{x^2}{2}\log 2x - \frac{1}{2}\int x\,dx$$

$$= \frac{x^2}{2}\log 2x - \frac{x^2}{4} + C.$$

उत्तर

प्रश्न 6. $x^2 \log x$.

हल : $\int x^2 \log x\,dx = \int (\log x).x^2\,dx$

खण्डशः समाकलन से,

$$\int uv\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\int v\,dx\right)dx$$

यहाँ $u = \log x$ तथा $v = x^2$ लेने पर

$$= (\log x)\int x^2 dx - \int\left[\frac{d}{dx}(\log x).\int x^2 dx\right]dx$$

$$= (\log x).\frac{x^3}{3} - \int \frac{1}{x}.\frac{x^3}{3}\,dx$$

$$= \frac{x^3}{3}\log x - \frac{1}{3}\int x^2 dx$$

$$= \frac{x^3}{3}\log x - \frac{x^3}{9} + C.$$

उत्तर

प्रश्न 7. $x \sin^{-1} x$.

हल : $\int x \sin^{-1} x\,dx = \int (\sin^{-1} x)\,x\,dx$

खण्डशः समाकलन से,

$$\int uv\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\int v\,dx\right)dx$$

यहाँ $u = \sin^{-1} x$ तथा $v = x$ लेने पर

$$= (\sin^{-1} x)\int x\,dx - \int\left[\frac{d}{dx}(\sin^{-1} x).\int x\,dx\right]dx$$

$$= (\sin^{-1} x)\frac{x^2}{2} - \int \frac{1}{\sqrt{1-x^2}} \cdot \frac{x^2}{2}\, dx$$

$$= \frac{x^2}{2}\sin^{-1} x + \frac{1}{2}\int \frac{1 - x^2 - 1}{\sqrt{1-x^2}}\, dx$$

$$= \frac{x^2}{2}\sin^{-1} x + \frac{1}{2}\int \left(\sqrt{1-x^2} - \frac{1}{\sqrt{1-x^2}}\right) dx$$

$$= \frac{x^2}{2}\sin^{-1} x + \frac{1}{2}\int \sqrt{1-x^2}\, dx - \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}\, dx$$

$$= \frac{x^2}{2}\sin^{-1} x + \frac{1}{2}\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x\right] - \frac{1}{2}\sin^{-1} x + C$$

$$\left[\because \int \sqrt{a^2 - x^2} = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{2}\right]$$

$$= \frac{x^2}{2}\sin^{-1} x + \frac{x}{4}\sqrt{1-x^2} - \frac{1}{4}\sin^{-1} x + C$$

$$= \frac{1}{4}(2x^2 - 1)\sin^{-1} x + \frac{x\sqrt{1-x^2}}{4} + C.$$

उत्तर

प्रश्न 8. $x \tan^{-1} x$.

हल : $\int x \tan^{-1} x\, dx = \int (\tan^{-1} x).x\, dx$

खण्डशः समाकलन करने पर,

$$= (\tan^{-1} x)\int x\, dx - \int\left(\frac{d}{dx}\tan^{-1} x \int x\, dx\right) dx$$

$$= (\tan^{-1} x)\frac{x^2}{2} - \int \frac{1}{1+x^2} \cdot \frac{x^2}{2}\, dx$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int \frac{x^2 + 1 - 1}{x^2 + 1}\, dx$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int \left(1 - \frac{1}{x^2 + 1}\right) dx$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int dx + \frac{1}{2}\int \frac{1}{1+x^2}\, dx$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}x + \frac{1}{2}\tan^{-1} x + C.$$

उत्तर

प्रश्न 9. $x \cos^{-1} x$.

हल : $\int x \cos^{-1} x\, dx = \int (\cos^{-1} x).x\, dx$

खण्डशः समाकलन करने पर,

$$= (\cos^{-1} x)\int x\, dx - \int\left[\frac{d}{dx}(\cos^{-1} x)\int x\, dx\right]dx$$

$$= \cos^{-1} x.\frac{x^2}{2} - \int -\frac{1}{\sqrt{1-x^2}}.\frac{x^2}{2}\, dx + C$$

मान लीजिए $x = \cos\theta$
तब $dx = -\sin\theta\, d\theta$

$$= \frac{x^2}{2}\cos^{-1} x + \frac{1}{2}\int\frac{\cos^2\theta}{\sqrt{1-\cos^2\theta}}[-\sin\theta]\, d\theta$$

$$= \frac{x^2}{2}\cos^{-1} x - \frac{1}{2}\int\frac{\cos^2\theta}{\sin\theta}\times\sin\theta\, d\theta$$

$$= \frac{x^2}{2}\cos^{-1} x - \frac{1}{4}\int(1+\cos 2\theta)\, d\theta$$

$$= \frac{x^2}{2}\cos^{-1} x - \frac{1}{4}\int 1\, d\theta - \frac{1}{4}\int\cos 2\theta\, d\theta$$

$$= \frac{x^2}{2}\cos^{-1} x - \frac{\theta}{4} - \frac{1}{4}.\frac{\sin 2\theta}{2} + C$$

$$= \frac{x^2}{2}\cos^{-1} x - \frac{\theta}{4} - \frac{1}{8}.2\sin\theta\cos\theta + C$$

$$= \frac{x^2}{2}\cos^{-1} x - \frac{\cos^{-1} x}{4} - \frac{\sqrt{1-x^2}}{4}.x + C$$

$$= \frac{x^2}{2}\cos^{-1} x - \frac{1}{4}\cos^{-1} x - \frac{x\sqrt{1-x^2}}{4} + C$$

$$= \frac{(2x^2-1)}{4}\cos^{-1} x - \frac{x\sqrt{1-x^2}}{4} + C$$

$$= (2x^2-1)\frac{\cos^{-1} x}{4} - \frac{x}{4}\sqrt{1-x^2} + C.$$ उत्तर

प्रश्न 10 $(\sin^{-1} x)^2$.

हल : $\int(\sin^{-1} x)^2 dx$

मान लीजिए $\sin^{-1} x = \theta$ हो, तब $x = \sin\theta$, $dx = \cos\theta\, d\theta$

$$= \int\theta^2\cos\theta\, d\theta$$

खण्डशः समाकलन करने पर,

$$= \theta^2\sin\theta - \int 2\theta\sin\theta\, d\theta$$

पुनः खण्डशः समाकलन करने पर

$$= \theta^2\sin\theta - 2[\theta(-\cos\theta) - \int 1(-\cos\theta)\, d\theta]$$

$$= \theta^2\sin\theta + 2\theta\cos\theta - 2\int\cos\theta\, d\theta$$

$$= \theta^2 \sin\theta + 2\theta\cos\theta - 2\sin\theta + C$$

$$= \theta^2 \sin\theta + 2\theta\sqrt{1-\sin^2\theta} - 2\sin\theta + C$$

$\sin\theta = x$ या $\theta = \sin^{-1} x$ रखने पर

$$= x(\sin^{-1} x)^2 + 2\sqrt{1-x^2}(\sin^{-1} x) - 2x + C$$

$$= (\sin^{-1} x)^2 x + 2\sqrt{1-x^2}\sin^{-1} x - 2x + C.$$ उत्तर

प्रश्न 11. $\dfrac{x\cos^{-1} x}{\sqrt{1-x^2}}$.

हल : $\dfrac{x\cos^{-1} x}{\sqrt{1-x^2}}$

मान लीजिए $\cos^{-1} x = t$ हो, तब, $x = \cos t$, $-\dfrac{1}{\sqrt{1-x^2}}dx = dt$

$$\therefore \quad \int \frac{x\cos^{-1} x}{\sqrt{1-x^2}}dx = -\int \cot t . t\, dt$$

$$= -[t(\sin t) - \int 1.\sin t\, dt]$$

$$= -t\sin t + (-\cos t) + C$$

$$= -\sqrt{1-\cos^2 t}.t - \cos t + C$$

$$= -\sqrt{1-x^2}\cos^{-1} x - x + C$$

$$= -\left[\sqrt{1+x^2}\cos^{-1} x + x\right] + C.$$ उत्तर

प्रश्न 12. $x \sec^2 x$.

हल : $\int x\sec^2 x\, dx$

खण्डशः समाकलन करने पर,

$$= x\tan x - \int 1.\tan x\, dx$$

$$= x\tan x + \log|\cos x| + C.$$ उत्तर

प्रश्न 13. $\tan^{-1} x$.

हल : $\int \tan^{-1} x\, dx = \int (\tan^{-1} x).1\, dx$

खण्डशः समाकलन करने पर,

$$= (\tan^{-1} x).x - \int \frac{1}{1+x^2}.x\, dx$$

$$= x\tan^{-1} x - \frac{1}{2}\int \frac{2x}{1+x^2}dx,$$ $1 + x^2 = t$ रखने पर

$$= x\tan^{-1} x - \frac{1}{2}\int \frac{dt}{t}$$ $\Rightarrow 2dx = dt$

$$= x\tan^{-1} x - \frac{1}{2}\log|t| + C$$

$$= x\tan^{-1} x - \frac{1}{2}\log|1+x^2| + C.$$ उत्तर

प्रश्न 14. $x (\log x)^2$.

हल : $$\int x(\log x)^2 dx = \int (\log x)^2 . x\, dx$$

खण्डशः समाकलन करने पर,

$$= (\log x)^2 . \frac{x^2}{2} - \int 2(\log x)\frac{1}{x}.\frac{x^2}{2} dx$$

$$= \frac{x^2}{2}(\log x)^2 - \int (\log x).x\, dx$$

पुनः खण्डशः समाकलन करने पर,

$$= \frac{x^2}{2}(\log x)^2 - \log (x)\frac{x^2}{2} + \int \frac{1}{x}.\frac{x^2}{2} dx$$

$$= \frac{x^2}{2}(\log x)^2 - \frac{x^2}{2}\log x + \frac{1}{2}\int x\, dx$$

$$= \frac{x^2}{2}(\log x)^2 - \frac{x^2}{2}\log x + \frac{1}{2}.\frac{x^2}{2} + C$$

$$= \frac{x^2}{2}(\log x)^2 - \frac{x^2}{2}(\log x) + \frac{x^2}{4} + C.$$ उत्तर

प्रश्न 15. $(x^2 + 1) \log x$.

हल : $$\int (x^2+1)\log x\, dx = \int (\log x)(x^2+1)\, dx$$

खण्डशः समाकलन करने पर,

$$= (\log x).\left(\frac{x^3}{3} + x\right) - \int \frac{1}{x}\left(\frac{x^3}{3} + x\right) dx$$

$$= \left(\frac{x^3}{3} + x\right)\log x - \int \left(\frac{x^2}{3} + 1\right) dx$$

$$= \left(\frac{x^3}{3} + x\right)\log x - \frac{1}{3}\int x^2 dx - \int 1\, dx$$

$$= \left(\frac{x^3}{3} + x\right)\log x - \frac{x^3}{9} - x + C.$$ उत्तर

प्रश्न 16. $e^x (\sin x + \cos x)$.

हल : $$\int e^x (\sin x + \cos x)\, dx$$

$$= \int \underset{\text{II}}{e^x} \underset{\text{I}}{\sin x}\, dx + \int e^x \cos x\, dx$$

$$= \sin x.e^x - \int \cos x.e^x dx + \int e^x \cos x\, dx$$

$$= \sin x.e^x + C.$$ उत्तर

प्रश्न 17. $\dfrac{xe^x}{(1+x)^2}$.

हल :
$$\int \frac{xe^x}{(1+x)^2}dx = \int \frac{e^x(x+1-1)}{(1+x)^2}dx$$
$$= \int e^x\left(\frac{1}{x+1} - \frac{1}{(x+1)^2}\right)dx$$

यदि $[f(x)] = \dfrac{1}{x+1}$ तो $\int e^x f(x) + f(x)\, dx$ यदि $f(x) = \dfrac{1}{(x+1)^2}$]
$$= \int \frac{1}{x+1}.e^x dx - \int \frac{1}{(x+1)^2}e^x dx$$
$$= \frac{1}{x+1}e^x - \int -\frac{1}{(x+1)^2}e^x dx - \int \frac{1}{(x+1)^2}e^x dx$$
$$= \frac{e^x}{x+1} + \int \frac{e^x}{(x+1)^2}dx - \int \frac{e^x}{(x+1)^2}dx$$
$$= \frac{e^x}{x+1} + C.$$
उत्तर

प्रश्न 18. $e^x\left(\dfrac{\mathbf{1+\sin x}}{\mathbf{1+\cos x}}\right)$.

हल :
$$\int e^x\left(\frac{1+\sin x}{1+\cos x}\right)dx = \int \frac{e^x\left(1+2\sin\frac{x}{2}\cos\frac{x}{2}\right)}{2\cos^2\frac{x}{2}}dx$$
$$= \int e^x\left(\frac{1}{2}\sec^2\frac{x}{2} + \tan\frac{x}{2}\right)dx$$
$$= \int e^x \tan\frac{x}{2}dx + \frac{1}{2}\int e^x \sec^2\frac{x}{2}dx \quad ....(i)$$

खण्डशः समाकलन करने पर,
$$\therefore \quad \int \tan\frac{x}{2}e^x dx = \left(\tan\frac{x}{2}\right).e^x - \int \frac{1}{2}\sec^2\frac{x}{2}e^x dx$$
$$= e^x \tan\frac{x}{2} - \frac{1}{2}\int \sec^2\frac{x}{2}e^x dx$$

यह मान समीकरण (i) में रखने पर
$$= e^x \tan\frac{x}{2} - \frac{1}{2}\int \sec^2\frac{x}{2}e^x dx + \frac{1}{2}\int e^x.\sec^2\frac{x}{2}dx$$
$$= e^x \tan\frac{x}{2} + C.$$
उत्तर

प्रश्न 19. $e^x\left(\dfrac{\mathbf{1}}{\mathbf{x}} - \dfrac{\mathbf{1}}{\mathbf{x^2}}\right)$.

हल :
$$\int e^x\left(\frac{1}{x} - \frac{1}{x^2}\right)dx = \int e^x.\frac{1}{x}dx - \int e^x.\frac{1}{x^2}dx$$

अब $e^x.\frac{1}{x}$ का खण्डशः समाकलन करने पर,

$$= \frac{1}{x}.e^x - \int \frac{-1}{x^2}.e^x dx - \int e^x.\frac{1}{x^2} dx$$

$$= \frac{1}{x}.e^x + \int \frac{1}{x^2}.e^x dx - \int e^x.\frac{1}{x^2} dx$$

$$= \frac{e^x}{x} + C$$ उत्तर

प्रश्न 20. $\frac{(x-3)e^x}{(x-1)^3}$.

हल : माना कि $I = \int \frac{(x-3)e^x}{(x-1)^3} dx = \int e^x \left[\frac{x-1-2}{(x-1)^3}\right] dx$

$$= \int e^x \left[\frac{1}{(x-1)^2} - \frac{2}{(x-1)^3}\right] dx$$

$$= \int \frac{1}{(x-1)^2}.e^x dx - \int \frac{2}{(x-1)^3}.\, e^x dx$$

यहाँ $e^x.\frac{1}{(x-1)^2} = t$ रखने पर

$$\left\{e^x.\frac{1}{(x-1)^2} + e^x.\left(\frac{-2}{(x-1)^3}\right)\right\} dx = dt$$

या $$e^x \left[\frac{1}{(x-1)^2} - \frac{2}{(x-1)^3}\right] dx = dt$$

$\therefore$ $$I = \int dt = t + C = \frac{e^x}{(x-1)^2} + C.$$ उत्तर

प्रश्न 21. $e^{2x} \sin x$.

हल : मान लीजिए $I = \int e^{2x} \sin x \, dx$

खण्डशः समाकलन करने पर,

$$I = e^{2x}(-\cos x) - \int 2e^{2x}.(-\cos x)\, dx$$

$$= -e^{2x} \cos x + 2\int e^{2x} \cos x \, dx$$

पुनः खण्डशः समाकलन करने पर

$$I = -e^{2x} \cos x + 2[e^{2x} \sin x - \int 2e^{2x} \sin x \, dx]$$

$$= -e^{2x} \cos x + 2e^{2x} \sin x - 4\int e^{2x} \sin x \, dx$$

$$I = -e^{2x} \cos x + 2e^{2x} \sin x - 41$$

या $$5I = e^{2x}.2 \sin x - e^{2x} \cos x$$

$\therefore$ $$I = \frac{1}{5} e^{2x}(2\sin x - \cos x) + C.$$ उत्तर

प्रश्न 22. $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

हल : $\int \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

मान लीजिए $x = \tan\theta$ हो, तब $dx = \sec^2\theta\, d\theta$

$$= \int \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\sec^2\theta\, d\theta$$
$$= \int \sin^{-1}(\sin 2\theta)\sec^2\theta\, d\theta$$
$$= \int 2\theta\sec^2\theta\, d\theta = 2\int\theta\sec^2\theta\, d\theta$$

खण्डशः समाकलन करने पर

$$= 2[\theta\tan\theta - \int 1.\tan\theta\, d\theta]$$
$$= 2[\theta\tan\theta + \log|\cos\theta|] + C$$

यहाँ $\tan\theta = x$, $\cos\theta = \frac{1}{\sqrt{1+x^2}}$ को रखने पर,

$$= 2[x\tan^{-1}x + \log\left|\frac{1}{\sqrt{1+x^2}}\right|] + C$$
$$= 2x\tan^{-1}x - 2.\frac{1}{2}\log(1+x^2) + C$$
$$= 2x\tan^{-1}x - \log(1+x^2) + C.$$

उत्तर

प्रश्न 23 व 24 में सही उत्तर का चयन कीजिए—

प्रश्न 23. $\int x^2 e^{x^3}dx$ **बराबर है—**

(A) $\frac{1}{3}e^{x^3} + C$ (B) $\frac{1}{3}e^{x^2} + C$

(C) $\frac{1}{2}e^{x^3} + C$ (D) $\frac{1}{2}e^{x^2} + C$

हल : $\int x^2e^{x^3}dx$

$x^3 = t$ रखने पर,

$\therefore \quad 3x^2dx = dt$

$$= \frac{1}{3}\int e^t dt = \frac{1}{3}e^t + C$$
$$= \frac{1}{3}e^{x^3}$$

अतः विकल्प (A) सही है।

उत्तर

प्रश्न 24. $\int e^x \sec x(1+\tan x)\,dx$ **बराबर है—**

(A) $e^x\cos x + C$ **(B)** $e^x\sec x + C$

(C) $e^x\sin x + C$ **(D)** $e^x\tan x + C$

हल : $\int e^x\sec x(1+\tan x)\,dx = \int e^x\sec x\,dx + \int e^x\sec x\tan x\,dx$

अब प्रथम समाकलन में खण्डशः समाकलन का प्रयोग करने पर,

$= \sec x \int e^x - \int \sec x \tan x \int e^x dx + \int e^x \sec x \tan x\, dx$

$= \sec x . e^x - \int e^x \sec x \tan x\, dx + \int e^x . \sec x \tan x\, dx$

$= e^x \sec x + C.$

अतः विकल्प (B) सही है। **उत्तर**

प्रश्नावली 7·7

प्रश्न 1 से 9 तक के प्रश्नों के फलनों का समाकलन कीजिए—

प्रश्न 1. $\sqrt{4 - x^2}$.

हल : $\int \sqrt{4 - x^2}\, dx$

$$= \int \sqrt{(2)^2 - x^2}\, dx$$

$$= \frac{x\sqrt{4 - x^2}}{2} + \frac{4}{2}\sin^{-1}\frac{x}{2} + C$$

$$\left[\because \int \sqrt{a^2 - x^2}\, dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} \text{ से}\right]$$

$$= \frac{x\sqrt{4 - x^2}}{2} + 2\sin^{-1}\frac{x}{2} + C$$

$$= \frac{1}{2}x\sqrt{4 - x^2} + 2\sin^{-1}\frac{x}{2} + C.$$ **उत्तर**

प्रश्न 2. $\sqrt{1 - 4x^2}$.

हल : $\int \sqrt{1 - 4x^2}\, dx = 2\int \sqrt{\frac{1}{4} - x^2}\, dx$

$$\therefore \quad = 2\left[\frac{x}{2}\sqrt{\frac{1}{4} - x^2} + \frac{1}{4.2}\sin^{-1}\left(\frac{x}{\frac{1}{2}}\right)\right] + C$$

$$\left[\because \int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= x\frac{\sqrt{1 - 4x^2}}{2} + \frac{1}{4}\sin^{-1} 2x + C$$

$$= \frac{x}{2}\sqrt{1 - 4x^2} + \frac{1}{4}\sin^{-1} 2x + C$$

$$= \frac{1}{4}\sin^{-1} 2x + \frac{1}{2}x\sqrt{1 - 4x^2} + C.$$ **उत्तर**

प्रश्न 3. $\sqrt{x^2+4x+6}$.

हल :
$$\int\sqrt{x^2+4x+6}\,dx = \int\sqrt{x^2+4x+4+2}\,dx = \int\sqrt{(x+2)^2+2}\,dx$$
$$= \frac{(x+2)\sqrt{(x+2)^2+2}}{2} + \frac{2}{2}\log\left|(x+2)+\sqrt{(x+2)^2+2}\right|+C$$
$$\left[\because \int\sqrt{x^2+a^2}\,dx = \frac{x\sqrt{x^2+a^2}}{2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| \text{ से}\right]$$
$$= \frac{1}{2}(x+2)\sqrt{x^2+4x+6} + \log\left|(x+2)+\sqrt{x^2+4x+6}\right|+C.$$ उत्तर

प्रश्न 4. $\sqrt{x^2+4x+1}$.

हल :
$$\int\sqrt{x^2+4x+1}\,dx = \int\sqrt{x^2+4x+4-3}\,dx$$
$$= \int\sqrt{(x+2)^2-3}\,dx$$
$$= \frac{(x+2)\sqrt{(x+2)^2-3}}{2} - \frac{3}{2}\log\left|\,(x+2)+\sqrt{(x+2)^2-3}\,\right|+C$$
$$\left[\because \int\sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|\right]$$
$$= \frac{1}{2}(x+2)\sqrt{x^2+4x+1} - \frac{3}{2}\log\left|\,(x+2)\sqrt{x^2+4x+1}\,\right|+C.$$ उत्तर

प्रश्न 5. $\sqrt{1-4x-x^2}$.

हल :
$$\int\sqrt{1-4x-x^2}\,dx = \int\sqrt{1-(x^2+4x+4)+4}\,dx$$
$$= \int\sqrt{5-(x+2)^2}\,dx$$
$$= \frac{1}{2}(x+2)\sqrt{5-(x+2)^2}+\frac{5}{2}\sin^{-1}\frac{x+2}{\sqrt{5}}+C$$
$$\left[\because \int\sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$
$$= \frac{1}{2}(x+2)\sqrt{1-4x-x^2}+\frac{5}{2}\sin^{-1}\frac{x+2}{\sqrt{5}}+C$$
$$= \frac{5}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right)+\frac{x+2}{2}\sqrt{1-4x-x^2}+C.$$ उत्तर

प्रश्न 6. $\sqrt{x^2+4x-5}$.

हल :
$$\int\sqrt{x^2+4x-5}\,dx = \int\sqrt{x^2+4x+4-9}\,dx$$
$$= \int\sqrt{(x+2)^2-9}\,dx$$
$$= \int\sqrt{(x+2)^2-(3)^2}\,dx$$
$$\left[\because \int\sqrt{x^2-a^2}\,dx = \int\frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|\right]$$
$$= \frac{1}{2}(x+2)\sqrt{(x+2)^2-9}$$
$$-\frac{9}{2}\log\left|(x+2)+\sqrt{(x+2)^2-9}\right|+C$$
$$= \frac{1}{2}(x+2)\sqrt{x^2+4x-5}$$
$$-\frac{9}{2}\log\left|(x+2)+\sqrt{x^2+4x-5}\right|+C.$$ उत्तर

प्रश्न 7. $\sqrt{1+3x-x^2}$.

हल :
$$\int\sqrt{1+3x-x^2}\,dx = \int\sqrt{1-\left(x^2-3x+\frac{9}{4}\right)+\frac{9}{4}}.dx$$
$$= \int\sqrt{\frac{13}{4}-\left(x-\frac{3}{2}\right)^2}\,dx$$
$$= \frac{1}{2}\left(x-\frac{3}{2}\right)\sqrt{\frac{13}{4}-\left(x-\frac{3}{2}\right)^2}+\frac{13}{8}\sin^{-1}\frac{x-\frac{3}{2}}{\sqrt{\frac{13}{4}}}+C$$
$$\left[\because \int\sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$
$$= \frac{2x-3}{4}\sqrt{1+3x-x^2}+\frac{13}{8}\sin^{-1}\left(\frac{2x-3}{\sqrt{13}}\right)+C.$$ उत्तर

प्रश्न 8. $\sqrt{x^2+3x}$.

हल :
$$\int\sqrt{x^2+3x}\,dx = \int\sqrt{x^2+3x+\frac{9}{4}-\frac{9}{4}}\,dx$$
$$= \int\sqrt{\left(x+\frac{3}{2}\right)^2-\frac{9}{4}}\,dx$$

$$= \frac{1}{2}\left(x+\frac{3}{2}\right)\sqrt{\left(x+\frac{3}{2}\right)^2-\frac{9}{4}}$$

$$-\frac{9}{8}\log\left|\left(x+\frac{3}{2}\right)+\sqrt{\left(x+\frac{3}{2}\right)^2-\frac{9}{4}}\right|+C$$

$$\left[\because \int\sqrt{x^2-a^2}\,dx = \int\frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2}{2}\log\left|\left(x+\sqrt{x^2-a^2}\right)\right|\right]$$

$$= \frac{2x+3}{4}\sqrt{x^2+3x}-\frac{9}{8}\log\left|x+\frac{3}{2}+\sqrt{x^2+3x}\right|+C. \quad \text{उत्तर}$$

प्रश्न 9. $\sqrt{1+\frac{x^2}{9}}$.

हल :

$$\int\sqrt{1+\frac{x^2}{9}}\,dx = \int\frac{\sqrt{x^2+9}}{3}\,dx$$

$$= \frac{1}{3}\int\sqrt{x^2+9}\,dx$$

$$\because \int\sqrt{x^2+a^2}\,dx = \int\frac{x\sqrt{x^2+a^2}}{2}+\frac{a}{2}\log|x+\sqrt{x^2+a^2}|$$

$$= \frac{1}{3}\left[\frac{x\sqrt{x^2+9}}{2}+\frac{3}{2}\log|x+\sqrt{x^2+9}|\right]+C$$

$$= \frac{1}{6}x\sqrt{x^2+9}+\frac{3}{2}\log|x+\sqrt{x^2+9}|+C. \quad \text{उत्तर}$$

प्रश्न 10 व 11 में सही उत्तर का चयन कीजिए—

प्रश्न 10. $\int\sqrt{1+x^2}\,dx$ **बराबर है—**

(A) $\frac{x}{2}\sqrt{1+x^2}+\frac{1}{2}\log|x+\sqrt{1+x^2}|+C$ (B) $\frac{2}{3}(1+x^2)^{3/2}+C$

(C) $\frac{2}{3}x(1+x^2)^{3/2}+C$

(D) $\frac{x^2}{2}\sqrt{1+x^2}+\frac{x^2}{2}\log|x+\sqrt{1+x^2}|+C$

हल :

$$\int\sqrt{1+x^2}\,dx = \int\sqrt{(1)^2+x^2}\,dx$$

सूत्रानुसार,

$$\int\sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\log|x+\sqrt{a^2+x^2}|+C$$

$$= \frac{x}{2}\sqrt{1+x^2}+\frac{1}{2}\log|x+\sqrt{1+x^2}|+C$$

अत: विकल्प (A) सही है। **उत्तर**

प्रश्न 11. $\int \sqrt{x^2 - 8x + 7}\, dx$ **बराबर है—**

(A) $\frac{1}{2}(x-4)\sqrt{x^2 - 8x + 7} + 9\log|x - 4 + \sqrt{x^2 - 8x + 7}| + C$

(B) $\frac{1}{2}(x+4)\sqrt{x^2 - 8x + 7} + 9\log|x + 4 + \sqrt{x^2 - 8x + 7}| + C$

(C) $\frac{1}{2}(x-4)\sqrt{x^2 - 8x + 7} - 3\sqrt{2}\log|x - 4 + \sqrt{x^2 - 8x + 7}| + C$

(D) $\frac{1}{2}(x-4)\sqrt{x^2 - 8x + 7} - \frac{9}{2}\log|x - 4 + \sqrt{x^2 - 8x + 7}| + C$

हल :

$$\int \sqrt{x^2 - 8x + 7}\,dx = \int \sqrt{x^2 - 8x + 16 + 7 - 16}\, dx$$

$$= \int \sqrt{(x-4)^2 - 9}\, dx$$

अब सूत्र $\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log|x + \sqrt{x^2 - a^2}| + C$ के प्रयोग से,

$$= \frac{x-4}{2}\sqrt{x^2 - 8x + 7} - \frac{9}{2}\log|(x-4) + \sqrt{x^2 - 8x + 7}| + C$$

अतः विकल्प (D) सही है। **उत्तर**

प्रश्नावली 7·8

योगों की सीमा के रूप में निम्नलिखित निश्चित समाकलनों का मान ज्ञात कीजिए—

प्रश्न 1. $\int_a^b x\, dx$

हल : यहाँ

$$f(x) = x \text{ और } b - a = nh$$

$$f(x) = x, f(a+h) = a + h, f(a + 2h) = a + 2h, ...,$$

$$f(a + \overline{n-1}h) = a + (n-1)h$$

$$\therefore \int_a^b x\, dx = \lim_{h \to 0} h[a + (a+h) + (a+2h) + (a+3h) + ... + (a + \overline{n-1}h)]$$

$$= \lim_{h \to 0} h[na + h(1 + 2 + 3 + ... + \overline{n-1})]$$

$$= \lim_{h \to 0} h\left[na + h \times \frac{n(n-1)}{2}\right]$$

$$= \lim_{h \to 0}\left[(nh)a + \frac{(nh)(nh-h)}{2}\right]$$

$$= \frac{(b-a)a + (b-a)(b-a-0)}{2} = \frac{2ab - 2a^2 + b^2 + a^2 - 2ab}{2}$$

$$= \frac{b^2 - a^2}{2}.$$ **उत्तर**

प्रश्न 2. $\int_0^5 (x+1)\, dx.$

हल : $\int_0^5 (x+1)\, dx$

दिया है :

$$f(x) = x + 1, a = 0, b = 5$$

$\therefore$

$$f(a) = a + 1, f(a + h) = a + h + 1$$

$$f(a + 2h) = a + 2h + 1$$

$$f(a + \overline{n-1}\, h)] = a + \overline{h-1}\, h + 1$$

$$nh = b - a = 5 - 0 = 5$$

$\therefore$

$$\int_0^5 (x+1)\, dx = \lim_{h \to 0} h(a+1) + (a+h+1) + (a+2h+1) + \ldots. + (a + \overline{n-1} + 1)]$$

$$= \lim_{h \to 0} h[n(a+1) + h(1 + 2 + 3 + \ldots. + \overline{n-1})]$$

$$= \lim_{h \to 0} h\left[n(a+1) + h.\frac{(n-1)n}{2}\right]$$

$$= \lim_{h \to \infty}\left[nh(a+1) + \frac{(nh-h)nh}{2}\right] \qquad \begin{bmatrix} a = 0 \text{ तथा} \\ nh = 5 \end{bmatrix}$$

$$= 5 \times 1 + \frac{5 \times 5}{2} = 5 + \frac{25}{2} = \frac{35}{2}.$$ **उत्तर**

प्रश्न 3. $\int_2^3 x^2\, dx$.

हल : $\int_2^3 x^2 dx$

यहाँ $nh = 3 - 2 = 1$ और $f(x) = x^2, a = 2, b = 3$

$$f(x) = x^2, f(2) = 2^2, f(2 + h) = (2 + h)^2$$

$$f(2 + 2h) = (2 + 2h)^2, \ldots. f(2 + \overline{n-1}\, h) = [2 + (n - 1)h]^2$$

$\because$

$$\int_a^b f(x)\, dx = \lim_{h \to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots. + f(a + \overline{n-1}\, h)]$$

$$\int_2^3 x^2\, dx = \lim_{h \to 0} h[2^2 + (2+h)^2 + (2+2h)^2 + (2+3h)^2 + \ldots.. + (2 + \overline{n-1}\, h)^2]$$

$$= \lim_{h \to 0} h\{2^2 + (2^2 + 2.2h + h^2) + [2^2 + 2(2.2h) + (2h)^2] + [2^2 + 2(2.3h) + (3h)^2] + \ldots.. + [2^2 + 2.2(n-1)h + (\overline{n-1}\, h)^2]\}$$

$$= \lim_{h \to 0} h\{[n.2^2 + 4h(1 + 2 + 3 + \ldots. + \overline{n-1}) + h^2[1 + 2^2 + 3^2 + \ldots. + (n-1)^2]\}$$

$$= \lim_{h \to 0}\left[nh.2^2 + 4h.\frac{(n-1)n}{2} + h^2\,\frac{(n-1)n(2n-1)}{6}\right]$$

$$= \lim_{h\to 0}\left[nh.4+\frac{4}{2}.(nh-h)nh+\frac{(nh-h)(nh)(2nh-h)}{6}\right]$$

$$= 1.4+2.1.1+\frac{1.1.2}{6} = 6+\frac{1}{3} = \frac{19}{3}.$$ उत्तर

प्रश्न 4. $\int_1^4 (x^2 - x)\, dx.$

हल : $\int_1^4 (x^2 - x)dx$

यहाँ $a = 1, b = 4, f(x) = x^2 - x, nh = b - a = 4 - 1 = 3$

$\therefore$

$$\int_1^4 (x^2 - x)\, dx = \lim_{h\to 0} h[f(1) + f(1 + h) + f(1 + 2h) + ... + f\{(1 + (n - 1)h\}]$$

$$= \lim_{h\to 0} h[(1 - 1) + \{(1 + h)^2 - (1 + h)\} + \{(1 + 2h)^2 - (1 + 2h)\} + ... + \{(1 + (n - 1)h)\}^2 - \{1 + (n - 1)h\}]$$

$$= \lim_{h\to 0} h[(1 + h^2 + 2h - 1 - h) + (1 + 4h^2 + 4h - 1 - 2h) + ... + \{1 + (n - 1)^2h^2 + 2(n - 1)h - 1 - (n - 1)h\}]$$

$$= \lim_{h\to 0} h[(h^2 + h) + (4h^2 + 2h) + ... + \{(n - 1)h^2 + (n - 1)h\}]$$

$$= \lim_{h\to 0} h[h^2(1^2 + 2^2 + 3^2 + ... + (n - 1)^2 + h\{1 + 2 + 3 + ... + (n - 1)\}]$$

$$= \lim_{h\to 0}\left[h^3.\frac{n(n-1)(2n-1)}{6}+h^2\,\frac{n(n-1)}{2}\right]$$

$$= \lim_{h\to 0}\left[\frac{nh(nh-h)(2nh-h)}{6}+\frac{nh(nh-h)}{2}\right]$$

$$= \frac{3.(3-0)(6-0)}{6}+\frac{3(3-0)}{2}$$

$$= 9+\frac{9}{2} = \frac{27}{2}.$$ उत्तर

प्रश्न 5. $\int_{-1}^1 e^x dx.$

हल : $\int_{-1}^1 e^x dx$

जबकि $\int_a^b f(x)\, dx = \lim_{h\to 0} h[f(a) + f(a+2h) + f(a+3h) + ... + (a+\overline{n-1}\,h)]$

दिया है : $a = -1, b = 1, b - a = 2 = nh$

$\therefore$

$$f(a) = e^x$$

$$f(-1) = e^{-1}, f(-1 + h) = e^{-1+h}$$

$$f(-1 + 2h) = e^{-1+2h}$$

$$f(-1+\overline{n-1}\,h) = e^{-1+\overline{n-1}\,h}$$

$\therefore \quad \int_{-1}^{1} e^x \, dx = \lim_{h\to 0} h[e^{-1} + e^{-1+h} + e^{-1-2h} + ... + e^{-1+\overline{n-1}h}]$

$= \lim_{h\to 0} he^{-1}[1 + e^h + e^{2h} + ... + e^{(n-1)h}]$

$= \lim_{h\to 0} he^{-1}\left[\frac{1-e^{nh}}{1-e^h}\right]$

$\left[\because a + ar + ar^2 + ... + ar^{n-1} = \frac{a(1-r^n)}{1-r} \text{ से}\right]$

$= \lim_{h\to 0} e^{-1} \frac{(1-e^{nh})}{\left(\frac{e^h-1}{h}\right)} = \frac{-e^{-1}(1-e^2)}{1}$

$\left[\because nh = 2, \lim_{h\to 0} \frac{e^h-1}{h} = 1\right]$

$= \frac{-1+e^2}{e} = e - \frac{1}{e}.$ **उत्तर**

प्रश्न 6. $\int_0^4 (x + e^{2x})\, dx$.

हल : हमें ज्ञात है :

$\int_a^b f(x)\, dx = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + ... + f(a+(n-1)h)]$

यहाँ $nh = b - a$

दिया है : $a = 0, b = 4, nh = 4 - 0 = 4$

$\therefore \quad \int_0^4 (x + e^{2x})\, dx = \lim_{h\to 0} h[f(0) + f(0+h) + f(0+2h) + ... + f(0+(n-1)h)]$

$= \lim_{h\to 0} h[(0+e^0) + (h + e^{2h}) + (2h + e^{2(2h)}) + ... + (n-1)h + e^{2(n-1)h}]$

$= \lim_{h\to 0} h[(1+2+3+....+(n-1)] + (1 + e^{2h} + e^{2(2h)} + ... + e^{2(n-1)h}]$

$= \lim_{h\to 0}\left[h^2.\frac{n(n-1)}{2} + h.\frac{1(1-e^{2nh})}{1-e^{2h}}\right]$

$= \lim_{h\to 0}\left[\frac{nh(nh-h)}{2} + \frac{1}{\frac{1-e^{2h}}{2h}.2}(1-e^{2nh})\right]$

$\left[\because \lim_{h\to 0}\frac{e^{2h}-1}{2h} = 1\right]$

$= \frac{4(4-0)}{2} - \frac{1}{2}(1-e^8) = 8 - \frac{1}{2}(1-e^8)$

$= \frac{15}{2} + \frac{e^8}{2} = \frac{1}{2}(15 + e^8).$ **उत्तर**

प्रश्नावली 7·9

प्रश्न 1 से 20 तक के प्रश्नों में निश्चित समाकलनों का मान ज्ञात कीजिए—

प्रश्न 1. $\int_{-1}^{1} (x+1)\, dx$.

हल : ज्ञात है :

$$\int_{-1}^{1} (x+1)\, dx = \left[\frac{x^2}{2} + x\right]_{-1}^{1} = \left(\frac{1}{2}+1\right) - \left(\frac{1}{2}-1\right)$$

$$= \frac{3}{2} + \frac{1}{2} = \frac{4}{2} = 2.$$ उत्तर

प्रश्न 2. $\int_{2}^{3} \frac{1}{x}\, dx$.

हल : ज्ञात है :

$$\int_{2}^{3} \frac{1}{x}\, dx = [\log x]_2^3 = \log 3 - \log 2 = \log \frac{3}{2}.$$ उत्तर

प्रश्न 3. $\int_{1}^{2} (4x^3 - 5x^2 + 6x + 9)\, dx$.

हल : ज्ञात है : $\int_{1}^{2} (4x^3 - 5x^2 + 6x + 9)\, dx$

$$= \left[4.\frac{x^4}{4} - 5.\frac{x^3}{3} + 6.\frac{x^2}{2} + 9x\right]_1^2$$

$$= \left[x^4 - \frac{5}{3}x^3 + 3x^2 + 9x\right]_1^2$$

$$= \left[\left(16 - \frac{5}{3}\times 8 + 3\times 4 + 9\times 2\right) - \left(1 - \frac{5}{3} + 3 + 9\right)\right]$$

$$= \left[16 - \frac{40}{3} + 12 + 18\right] - \left[13 - \frac{5}{3}\right]$$

$$= \left[46 - \frac{40}{3}\right] - \left[13 - \frac{5}{3}\right] = 33 - \frac{40}{3} + \frac{5}{3}$$

$$= 33 - \frac{35}{3} = \frac{99-35}{3} = \frac{64}{3}.$$ उत्तर

प्रश्न 4. $\int_{0}^{\pi/4} \sin 2x\, dx$.

हल : ज्ञात है :

$$\int_{0}^{\pi/4} \sin 2x\, dx = -\frac{1}{2}[\cos 2x]_0^{\pi/4}$$

$$= -\frac{1}{2}[0-1] = \frac{1}{2}.$$ उत्तर

प्रश्न 5. $\int_{0}^{\pi/2} \cos 2x\, dx$.

हल : ज्ञात है :

$$\int_0^{\pi/2} \cos 2x\, dx = \frac{1}{2}[\sin 2x]_0^{\pi/2} = \frac{1}{2}[\sin 2 \times \frac{\pi}{2} - \sin 0]$$

$$= \frac{1}{2}[\sin \pi - 0] = \frac{1}{2}(0-0) = 0.$$ **उत्तर**

प्रश्न 6. $\int_4^5 e^x\, dx.$

हल : ज्ञात है :

$$\int_4^5 e^x dx = [e^x]_4^5 = e^5 - e^4 = e^4(e-1).$$ **उत्तर**

प्रश्न 7. $\int_0^{\pi/4} \tan x\, dx.$

हल : ज्ञात है :

$$\int_0^{\pi/4} \tan x\, dx = [\log \cos x]_0^{\pi/4}$$

$$= -\left[\log \cos \frac{\pi}{4} - \log \cos 0\right]$$

$$= -\left[\log \frac{1}{\sqrt{2}} - 0\right] = \log \sqrt{2} = \log (2)^{1/2} = \frac{1}{2}\log 2.$$ **उत्तर**

प्रश्न 8. $\int_{\pi/6}^{\pi/4} \operatorname{cosec} x\, dx.$

हल : ज्ञात है :

$$\int_{\pi/6}^{\pi/4} \operatorname{cosec} x\, dx = [\log\ (\operatorname{cosec} x - \cot x]_{\pi/6}^{\pi/4}$$

$$= \log\left(\operatorname{cosec}\frac{\pi}{4} - \cot\frac{\pi}{4}\right) - \log\left(\operatorname{cosec}\frac{\pi}{6} - \cot\frac{\pi}{6}\right)$$

$$= \log\ (\sqrt{2}-1) - \log\ (2-\sqrt{3})$$

$$= \log\left(\frac{\sqrt{2}-1}{2-\sqrt{3}}\right).$$ **उत्तर**

प्रश्न 9. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}.$

हल : ज्ञात है :

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = [\sin^{-1} x]_0^1 = \sin^{-1} 1 - \sin^{-1} 0$$

$$= \sin^{-1}\left(\sin\frac{\pi}{2}\right) - \sin^{-1}(\sin 0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$ **उत्तर**

प्रश्न 10. $\int_0^1 \frac{dx}{1+x^2}.$

हल : हमें ज्ञात है :

$$\int_0^1 \frac{dx}{1+x^2} = \left|\tan^{-1} x\right|_0^1 = \tan^{-1} 1 - \tan^{-1} 0$$

$$= \tan^{-1}\left(\tan\frac{\pi}{4}\right) - \tan^{-1}(\tan 0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}.$$ **उत्तर**

प्रश्न 11. $\int_2^3 \frac{dx}{x^2-1}$.

हल : ज्ञात है :

$$\int_2^3 \frac{dx}{x^2-1} = \frac{1}{2}\left[\log\frac{x-1}{x+1}\right]_2^3 = \frac{1}{2}\left[\log\frac{2}{4} - \log\frac{1}{3}\right]$$

$$= \frac{1}{2}\left[\log\left(\frac{1}{2}\times\frac{3}{1}\right)\right] = \frac{1}{2}\log\frac{3}{2}.$$ **उत्तर**

प्रश्न 12. $\int_0^{\pi/2} \cos^2 x\, dx$.

हल : मान लीजिए

$$I = \int_0^{\pi/2} \cos^2 x\, dx \quad ...(i)$$

$$= \int_0^{\pi/2} \cos^2\left(\frac{\pi}{2} - x\right) dx \quad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$I = \int_0^{\pi/2} \sin^2 x\, dx \quad ...(ii)$$

समीकरण (i) तथा (ii) को जोड़ने पर,

$$2I = \int_0^{\pi/2} \cos^2 x\, dx + \int_0^{\pi/2} \sin^2 x\, dx$$

$$= \int_0^{\pi/2} (\cos^2 x + \sin^2 x)\, dx$$

$$= \int_0^{\pi/2} 1\, dx = [x]_0^{\pi/2}$$

$$2I = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$\therefore \quad I = \frac{\pi}{4}.$$ **उत्तर**

प्रश्न 13. $\int_2^3 \frac{x\, dx}{x^2+1}$.

हल : ज्ञात है : $\int_2^3 \frac{x\,dx}{x^2+1}$ मान लीजिए $x^2 + 1 = t$ हो, तब $\therefore$ $2x\,dx = dt$

$$\int \frac{x}{x^2+1}dx = \frac{1}{2}\int\frac{dt}{t} = \frac{1}{2}\log t = \frac{1}{2}\log(x^2+1)$$

$$\therefore \quad \int_2^3 \frac{x\,dx}{x^2+1} = \frac{1}{2}\left[\log(x^2+1)\right]_2^3$$

$$= \frac{1}{2}[\log(3^2+1) - \log(2^2+1)]$$

$$= \frac{1}{2}[\log 10 - \log 5]$$

$$= \frac{1}{2}\log\frac{10}{5} = \frac{1}{2}\log 2.$$ **उत्तर**

प्रश्न 14. $\int_0^1 \frac{2x+3}{5x^2+1}dx.$

हल : ज्ञात है : $\int_0^1 \frac{2x+3}{5x^2+1}dx$

$$\int \frac{2x+3}{5x^2+1}dx = \frac{1}{5}\int \frac{10x}{5x^2+1}dx + 3\int \frac{1}{5x^2+1}dx$$

$$= \frac{1}{5}\int \frac{10x}{5x^2+1}dx + \frac{3}{5}\int \frac{1}{x^2+\frac{1}{5}}dx$$

प्रथम समाकलन में माना $5x^2 + 1 = t$ तब $\therefore\ 10x\ dx = dt$

$$= \frac{1}{5}\int \frac{dt}{t} + \frac{3}{5}\int \frac{1}{(x^2)+\left(\frac{1}{\sqrt{5}}\right)^2}dx$$

$$= \frac{1}{5}\log t + \frac{3}{5}\times \sqrt{5}\ \tan^{-1}\left(\frac{x}{1/\sqrt{5}}\right)$$

$$= \frac{1}{5}\log\ (5x^2+1) + \frac{3}{\sqrt{5}}\ \tan^{-1}(x\sqrt{5})$$

$$\therefore \quad \int_0^1 \frac{2x+3}{5x^2+1}dx = \left[\frac{1}{5}\log\ (5x^2+1) + \frac{3}{\sqrt{5}}\tan^{-1}(\sqrt{5}x)\right]_0^1$$

$$= \left[\frac{1}{5}\log\ (5+1) + \frac{3}{\sqrt{5}}\tan^{-1}(\sqrt{5})\right] - \left[\frac{1}{5}\log\ (0+1) + \frac{3}{\sqrt{5}}\tan^{-1} 0\right]$$

$$= \frac{1}{5}\log 6 + \frac{3}{\sqrt{5}}\tan^{-1}\sqrt{5}.$$

$(\because \log 1 = 0)$ **उत्तर**

प्रश्न 15. $\int_0^1 xe^{x^2}dx.$

हल : $\int xe^{x^2}dx$, मान लीजिए $x^2 = t$ तब $2x\ dx = t\ dt$

$$= \frac{1}{2}\int e^t dt = \frac{1}{2}e^t = \frac{1}{2}e^{x^2}$$

$$\therefore \quad \int_0^1 xe^{x^2}dx = \left|\frac{1}{2}e^{x^2}\right|_0^1 = \frac{1}{2}e^1 - \frac{1}{2}e^0 = \frac{1}{2}(e-1).$$

उत्तर

प्रश्न 16. $\int_1^2 \frac{5x^2}{x^2+4x+3}dx.$

हल : $\int \frac{5x^2}{x^2+4x+3}dx$

$5x^2$ को $x^2 + 4x + 3$ से भाग देने पर,

$$\therefore \quad \frac{5x^2}{x^2+4x+3} = 5 - \frac{20x+15}{x^2+4x+3}$$

अब $$\frac{20x+15}{x^2+4x+3} = \frac{20x+15}{(x+3)(x+1)} = \frac{A}{x+3} + \frac{B}{x+1}$$

$$\therefore \quad 20x + 15 = A(x + 1) + B(x + 3)$$

$x = -3$ लेने पर,

$$-60 + 15 = A(-2)$$

$$A = \frac{45}{2}$$

तथा $x = -1$ लेने पर,

$$-20 + 15 = B \times 2$$

$$\therefore \quad B = -\frac{5}{2}$$

$$\therefore \quad \frac{20x+15}{x^2+4x+3} = \frac{45}{2(x+3)} - \frac{5}{2(x+1)}$$

अब $$\int \frac{5x^2}{x^2+4x+3}dx = \int\left[5 - \frac{45}{2(x+3)} + \frac{5}{2(x+1)}\right]dx$$

$$= 5x - \frac{45}{2}\log|x+3| + \frac{5}{2}\log|x+1|$$

$$\therefore \quad \int_1^2 \frac{5x^2}{x^2+4x+3}dx = \left[5x - \frac{45}{2}\log|x+3| + \frac{5}{2}\log|x+1|\right]_1^2$$

$$= \left[\left(10 - \frac{45}{2}\log 5 + \frac{5}{2}\log 3\right) - \left(5 - \frac{45}{2}\log 4 + \frac{5}{2}\log 2\right)\right]$$

$$= 5 - \frac{45}{2}\log\frac{5}{4} + \frac{5}{2}\log\frac{3}{2}$$

$$= 5 - \frac{5}{2}\left(9\log\frac{5}{4} - \log\frac{3}{2}\right).$$ उत्तर

प्रश्न 17. $\int_0^{\pi/4} (2\sec^2 x + x^3 + 2)\, dx.$

हल : ज्ञात है :

$$\int_0^{\pi/4} (2\sec^2 x + x^3 + 2)\, dx = 2\int_0^{\pi/4} \sec^2 x\, dx + \int_0^{\pi/4} x^3 dx + 2\int_0^{\pi/4} dx$$

$$= \left[2\tan x + \frac{x^4}{4} + 2x\right]_0^{\pi/4}$$

$$= \left(2\tan\frac{\pi}{4} + \frac{\pi^4}{1024} + \frac{\pi}{2}\right) - 0$$

$$= 2 + \frac{\pi^4}{1024} + \frac{\pi}{2}$$

$$= \frac{\pi^4}{1024} + \frac{\pi}{2} + 2.$$ उत्तर

प्रश्न 18. $\int_0^{\pi}\left(\sin^2\frac{x}{2}-\cos^2\frac{x}{2}\right)dx.$

हल : ज्ञात है :

$$\int_0^{\pi}\left(\sin^2\frac{x}{2}-\cos^2\frac{x}{2}\right)dx = -\int_0^{\pi}\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)dx$$

$$= -\int_0^{\pi}\cos x\,dx = -[\sin x]_0^{\pi}$$

$$= -[\sin\pi+\sin 0] = 0.$$

उत्तर

प्रश्न 19. $\int_0^{2}\frac{6x+3}{x^2+4}dx.$

हल : ज्ञात है :

$$\int_0^{2}\frac{6x+3}{x^2+4}dx = 3\int_0^{2}\frac{2x}{x^2+4}dx+3\int_0^{2}\frac{dx}{x^2+4}$$

$$= \left[3\log|x^2+4|+3\times\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^2$$

$$= \left(3\log 8+\frac{3}{2}\tan^{-1}\frac{2}{2}\right)-\left(3\log 4+\frac{3}{2}\tan^{-1}0\right)$$

$$= 3\log\frac{8}{4}+\frac{3}{2}\tan^{-1}1$$

$$= 3\log 2+\frac{3}{2}.\frac{\pi}{4}$$

$$= 3\log 2+\frac{3\pi}{8}.$$

उत्तर

प्रश्न 20. $\int_0^{1}\left(xe^x+\sin\frac{\pi x}{4}\right)dx.$

हल : ज्ञात है :

$$\int_0^{1}\left(xe^x+\sin\frac{\pi x}{4}\right)dx = \int_0^{1}xe^x dx+\int_0^{1}\sin\frac{\pi x}{4}dx$$

$$= \left[xe^x\right]_0^1-\int_0^1 1.e^x dx+\int_0^1\sin\frac{\pi x}{4}dx$$

$$= [1.e-0]-\left[e^x\right]_0^1-\frac{4}{\pi}\left[\cos\pi\frac{x}{4}\right]_0^1$$

$$= e-(e-1)-\frac{4}{\pi}\left(\cos\frac{\pi}{4}-\cos 0\right)$$

$$= 1-\frac{4}{\pi}\left(\frac{1}{\sqrt{2}}-1\right)$$

$$= 1 - \frac{2\sqrt{2}}{\pi} + \frac{4}{\pi}$$

$$= 1 + \frac{4}{\pi} - \frac{2\sqrt{2}}{\pi}.$$ उत्तर

प्रश्न 21 एवं 22 में सही उत्तर का चयन कीजिए—

प्रश्न 21. $\int_1^{\sqrt{3}} \frac{dx}{1+x^2}$ बराबर है :

(A) $\frac{\pi}{3}$ (B) $\frac{2\pi}{3}$

(C) $\frac{\pi}{6}$ (D) $\frac{\pi}{12}$

हल :

$$\int_1^{\sqrt{3}} \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_1^{\sqrt{3}}$$

$$= \tan^{-1}(\sqrt{3}) - \tan^{-1}(1)$$

$$= \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

अत: विकल्प (D) सही है। उत्तर

प्रश्न 22. $\int_0^{2/3} \frac{dx}{4+9x^2}$ बराबर है :

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{12}$

(C) $\frac{\pi}{24}$ (D) $\frac{\pi}{4}$

हल :

$$\int_0^{2/3} \frac{dx}{4+9x^2} = \frac{1}{9}\int_0^{2/3} \frac{dx}{\frac{4}{9}+x^2}$$

$$= \frac{1}{9}\int_0^{2/3} \frac{dx}{\left(\frac{2}{3}\right)^2 + x^2}$$

$$= \frac{1}{9} \cdot \frac{1}{2/3}\left[\tan^{-1} \frac{x}{2/3}\right]_0^{2/3}$$

$$= \frac{1}{9} \times \frac{3}{2}\left[\tan^{-1} \frac{3x}{2}\right]_0^{2/3}$$

$$= \frac{1}{6}\left[\tan^{-1} \frac{3}{2} \times \frac{2}{3} - \tan^{-1} 0\right]$$

$$= \frac{1}{6}\left[\tan^{-1}(1) - \tan^{-1}(0)\right] = \frac{1}{6}\left[\frac{\pi}{4} - 0\right] = \frac{\pi}{24}$$

अत: विकल्प (C) सही है। उत्तर

प्रश्नावली 7.10

प्रश्न 1 से 8 तक के प्रश्नों में समाकलनों का मान प्रतिस्थापन का उपयोग करते हुए ज्ञात कीजिए—

प्रश्न 1. $\int_0^1 \frac{x}{x^2+1}dx$

हल : दिया है : $\int_0^1 \frac{x}{x^2+1}dx = \frac{1}{2}\int_0^1 \frac{2x}{x^2+1}dx$

मान लीजिए $x^2 + 1 = t$ हो, तब $\therefore 2x\, dx = dt$

अत: $$\frac{1}{2}\int_0^1 \frac{2x}{x^2+1}dx = \frac{1}{2}\int_1^2 \frac{dt}{t} = \frac{1}{2}[\log t]_1^2$$

$$= \frac{1}{2}(\log 2 - \log 1)$$

$$= \frac{1}{2}\log 2.$$ उत्तर

प्रश्न 2. $\int_0^{\pi/2} \sqrt{\sin\phi}\cos^5\phi\, d\phi$

हल : $$\int_0^{\pi/2} \sqrt{\sin\phi}\cos^5\phi\, d\phi = \int_0^{\pi/2} \sqrt{\sin\phi}\cos^4\phi\cos\phi\, d\phi$$

$$= \int_0^{\pi/2} \sqrt{\sin\phi}\,(1-\sin^2\phi)^2\cos\phi\, d\phi$$

मान लीजिए $\sin\phi = t$ हो, तो $\cos\phi\, d\phi = dt$ यदि $\phi = \frac{\pi}{2}, t = 1$ तथा यदि $\phi = 0, t = 0$

$$= \int_0^1 \sqrt{t}\,(1-t^2)^2 dt = \int_0^1 \sqrt{t}\,(1-2t^2+t^4)dt$$

$$= \int_0^1 (\sqrt{t} - 2t^{5/2} + t^{9/2})dt$$

$$= \left[\frac{2}{3}t^{3/2} - 2.\frac{2}{7}t^{7/2} + \frac{2}{11}t^{11/2}\right]_0^1$$

$$= \frac{2}{3} - \frac{4}{7} + \frac{2}{11} = \frac{154-132+42}{231} = \frac{64}{231}.$$ उत्तर

प्रश्न 3. $\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

हल : $\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

मान लीजिए $x = \tan\theta$ हो, तब $\therefore dx = \sec^2\theta\, d\theta$

यदि $x = 1, \theta = \frac{\pi}{4}$ तथा यदि $x = 0; \theta = 0$

$$= \int_0^{\pi/4} \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\sec^2\theta\, d\theta$$

$$= \int_0^{\pi/4} \sin^{-1}(\sin 2\theta)\sec^2\theta\, d\theta$$

$$= \int_0^{\pi/4} 2\theta \sec^2\theta\, d\theta = 2\int_0^{\pi/4} \theta \sec^2\theta\, d\theta$$

अब खण्डशः समाकलन करने पर,

$$I = 2[\theta.\tan\theta]_0^{\pi/4} - 2\int_0^{\pi/4} 1.\tan\theta\, d\theta$$

$$= 2\left[\frac{\pi}{4}\tan\frac{\pi}{4} - 0\right] + 2\left[|\log\cos\theta|_0^{\pi/4}\right]$$

$$= 2\left[\frac{\pi}{4} + \log\cos\frac{\pi}{4} - \log\cos 0\right]$$

$$= 2\left[\frac{\pi}{4} + \log\frac{1}{\sqrt{2}} - \log 1\right]$$

$$= \frac{\pi}{2} - 2\log\sqrt{2} = \frac{\pi}{2} - 2.\frac{1}{2}\log 2 = \frac{\pi}{2} - \log 2.$$ उत्तर

प्रश्न 4. $\int_0^2 x\sqrt{x+2}\, dx$

हल : $\int_0^2 x\sqrt{x+2}\, dx$, मान लीजिए $x + 2 = t^2$ हो, तब $dx = 2t\, dt$

$x = t^2 - 2$, यदि $x = 2, t = 2$, यदि $x = 0, t = \sqrt{2}$

$$= \int_{\sqrt{2}}^2 (t^2 - 2)t.2t\, dt = 2\int_{\sqrt{2}}^2 (t^4 - 2t^2)\, dt$$

$$= 2\left[\frac{t^5}{5} - 2.\frac{t^3}{3}\right]_{\sqrt{2}}^2$$

$$= 2\left[\left(\frac{32}{5} - \frac{2}{3}\times 8\right) - \left(\frac{(\sqrt{2})^5}{5} - \frac{2}{3}(\sqrt{2})^3\right)\right]$$

$$= 2\left[\left(\frac{32}{5} - \frac{16}{3}\right) - \left(\frac{4\sqrt{2}}{5} - \frac{4\sqrt{2}}{3}\right)\right]$$

$$= \frac{64}{5} - \frac{32}{3} - \frac{8\sqrt{2}}{5} + \frac{8\sqrt{2}}{3}$$

$$= \frac{192-160}{15} + \frac{8\sqrt{2}}{15}(5-3)$$

$$= \frac{32}{15} + \frac{16\sqrt{2}}{15} = \frac{16}{15}(2+\sqrt{2}) = \frac{16\sqrt{2}}{15}(\sqrt{2}+1).$$

उत्तर

प्रश्न 5. $\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}\, dx$

हल : $\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}\, dx$

मान लीजिए $\cos x = t$ हो, तब $\therefore -\sin x\, dx = dt$

जब $x = \frac{\pi}{2}, t = 0,$ और जब $x = 0, t = 1$

$$= -\int_1^0 \frac{dt}{1+t^2} = \int_0^1 \frac{dt}{1+t^2} = |\tan^{-1} t|_0^1$$

$$= \tan^{-1} 1 - 0 = \frac{\pi}{4}.$$ उत्तर

प्रश्न 6. $\int_0^2 \frac{dx}{x+4-x^2}$

हल :

$$\int_0^2 \frac{dx}{x+4-x^2} = \int_0^2 \frac{dx}{4-\left(x^2-x+\frac{1}{4}\right)+\frac{1}{4}}$$

$$= \int_0^2 \frac{dx}{\frac{17}{4}-\left(x-\frac{1}{2}\right)^2} \qquad \left[\because \int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\frac{a+x}{a-x} \text{ से}\right]$$

$$= \frac{1}{2.\frac{\sqrt{17}}{2}}\left(\log\frac{\frac{\sqrt{17}}{2}+\left(x-\frac{1}{2}\right)}{\frac{\sqrt{17}}{2}-\left(x-\frac{1}{2}\right)}\right)_0^2$$

$$= \frac{1}{\sqrt{17}}\left[\log\left(\frac{\frac{\sqrt{17}}{2}+\frac{3}{2}}{\frac{\sqrt{17}}{2}-\frac{3}{2}}\right) - \log\left(\frac{\frac{\sqrt{17}}{2}-\frac{1}{2}}{\frac{\sqrt{17}}{2}+\frac{1}{2}}\right)\right]$$

$$= \frac{1}{\sqrt{17}}\left[\log\frac{\sqrt{17}+3}{\sqrt{17}-3} - \log\frac{\sqrt{17}-1}{\sqrt{17}+1}\right]$$

$$= \frac{1}{\sqrt{17}}\log\left(\frac{\sqrt{17}+3}{\sqrt{17}-3}\times\frac{\sqrt{17}+1}{\sqrt{17}-1}\right)$$

$$= \frac{1}{\sqrt{17}}\log\left(\frac{20+4\sqrt{17}}{20-4\sqrt{17}}\right) = \frac{1}{\sqrt{17}}\log\left(\frac{5+\sqrt{17}}{5-\sqrt{17}}\right)$$

$$= \frac{1}{\sqrt{17}}\log\left(\frac{5+\sqrt{17}}{5-\sqrt{17}}\times\frac{5+\sqrt{17}}{5+\sqrt{17}}\right)$$

$$= \frac{1}{\sqrt{17}}\log\frac{(5+\sqrt{17})^2}{25-17}$$

$$= \frac{1}{\sqrt{17}}\log\left(\frac{42+10\sqrt{17}}{8}\right) = \frac{1}{\sqrt{17}}\log\left(\frac{21+5\sqrt{17}}{4}\right).$$ उत्तर

प्रश्न 7. $\int_{-1}^{1} \frac{dx}{x^2 + 2x + 5}$

हल : $$\int_{-1}^{1} \frac{dx}{x^2+2x+5} = \int_{-1}^{1} \frac{dx}{(x+1)^2+4}$$

$\because$ $$\int \frac{1}{x^2+a^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$= \frac{1}{2}\left(\tan^{-1}\frac{x+1}{2}\right)_{-1}^{1}$$

$$= \frac{1}{2}[\tan^{-1}1 - \tan^{-1}0]$$

$$= \frac{1}{2}\left[\tan^{-1}\left(\tan\frac{\pi}{4}\right) - 0\right]$$

$$= \frac{1}{2} \times \frac{\pi}{4} = \frac{\pi}{8}.$$ उत्तर

प्रश्न 8. $\int_{1}^{2} \left(\frac{1}{x} - \frac{1}{2x^2}\right) e^{2x} dx$

हल : $$\int_{1}^{2} \left(\frac{1}{x} - \frac{1}{2x^2}\right) e^{2x} dx$$

$$= \int_{1}^{2} \left(e^{2x}.\frac{1}{x} - e^{2x}.\frac{1}{2x^2}\right) dx$$

$$= \int_{1}^{2} \frac{e^{2x}}{x} dx - \int_{1}^{2} \frac{e^{2x}}{2x^2} dx$$

प्रथम भाग का खण्डशः समाकलन करने पर,

$$= \left[\frac{1}{x}.\frac{e^{2x}}{2}\right]_{1}^{2} - \int_{1}^{2} \left(-\frac{1}{x^2}\right).\frac{e^{2x}}{2} dx - \int_{1}^{2} \frac{e^{2x}}{2x^2} dx$$

$$= \left(\frac{e^4}{4} - \frac{e^2}{2}\right) + \int_{1}^{2} \frac{e^{2x}}{2x^2} dx - \int_{1}^{2} \frac{e^{2x}}{2x^2} dx$$

$$= \left(\frac{e^4}{4} - \frac{e^2}{4}\right) = \frac{e^2}{4}(e^2 - 2).$$ उत्तर

प्रश्न 9 एवं 10 में सही उत्तर का चयन कीजिए।

प्रश्न 9. समाकलन $\int_{1/3}^{1} \frac{(x - x^3)^{1/3}}{x^4} dx$ **का मान है :**

(A) 6 (B) 0 (C) 3 (D) 4

हल : $$\int_{1/3}^{1} \frac{(x-x^3)^{1/3}}{x^4} dx = \int_{1/3}^{1} \frac{(x^3)^{1/3}\left(\frac{x}{x^3} - 1\right)^{1/3}}{x^4} dx$$

$$= \int_{1/3}^{1} \frac{\left(\frac{1}{x^2}-1\right)^{1/3}}{x^3} dx$$

मान लीजिए $\frac{1}{x^2} = t$ हो, तब $\frac{-2}{x^3} dx = dt$

जब $x = 1$ हो, तब $t = 1$ और जब $x = \frac{1}{3}$ हो, तब $t = 9$.

$$= -\frac{1}{2}\int_9^1 (t-1)^{1/3} dt$$

$$= -\frac{1}{2}\left[\frac{(t-1)^{4/3}}{4/3}\right]_9^1$$

$$= -\frac{1}{2}\times\frac{3}{4}\left[(t-1)^{4/3}\right]_9^1$$

$$= -\frac{3}{8}\left[0-(8)^{4/3}\right] = -\frac{3}{8}[-2^{3\times\frac{4}{3}}]$$

$$= -\frac{3}{8}(-16) = 6$$

अतः विकल्प (A) सही है। **उत्तर**

प्रश्न 10. यदि $f(x) = \int_0^x t \sin t\, dt$, तब $f'(x)$ है—

(A) $\cos x + x \sin x$ (B) $x \sin x$
(C) $x \cos x$ (D) $\sin x + x \cos x$

हल : ∵ $f(x) = \int_0^x t \sin t\, dt$

∵ $\int t \sin t\, dt = [t(-\cos t) - \int t\cdot(-\cos t)\, dt$
$= -t \cos t + \sin t$

अब $\int_0^x t \sin t\, dt = \left[-t \cos t\right]_0^x + \left[\sin t\right]_0^x = -x \cos x + \sin x$

∴ $f'(x) = -[1 (\cos x) - x \sin x] + \cos x$
$= -\cos x + x \sin x + \cos x = x \sin x$

अतः विकल्प (B) सही है। **उत्तर**

प्रश्नावली 7.11

प्रश्न निश्चित समाकलनों के गुणधर्मों का उपयोग करते हुए 1 से 19 तक के प्रश्नों में समाकलनों का मान ज्ञात कीजिए।

प्रश्न 1. $\int_0^{\pi/2} \cos^2 x\, dx$

हल : मान लीजिए $I = \int_0^{\pi/2} \cos^2 x\, dx$ ....(i)

$$= \int_0^{\pi/2} \cos^2\left(\frac{\pi}{2}-x\right) dx \quad \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx \text{ से}\right]$$

$$I = \int_0^{\pi/2} \sin^2 x\, dx \qquad ...(ii)$$

समीकरण (i) तथा (ii) को जोड़ने पर,

$$2I = \int_0^{\pi/2} \cos^2 x\, dx + \int_0^{\pi/2} \sin^2 x\, dx$$

$$= \int_0^{\pi/2} (\cos^2 x + \sin^2 x) dx = \int_0^{\pi/2} 1\, dx = [x]_0^{\pi/2}$$

$$= \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$I = \frac{\pi}{4}.$$ **उत्तर**

प्रश्न 2. $\displaystyle\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$

हल : मान लीजिए

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \qquad ...(i)$$

$$= \int_0^{\pi/2} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}} dx$$

$$\left[\because \int_0^x f(x)\, dx = \int_a^x f(a - x) dx\right]$$

$$= \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx \qquad ...(ii)$$

समीकरण (i) तथा (ii) को जोड़ने पर,

$$2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx + \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx$$

$$= \int_0^{\pi/2} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

$$= \int_0^{\pi/2} 1\, dx = [x]_1^{\pi/2} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$\therefore \qquad I = \frac{\pi}{4}.$$ **उत्तर**

प्रश्न 3. $\displaystyle\int_0^{\pi/2} \frac{\sin^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x} dx$

हल : मान लीजिए

$$I = \int_0^{\pi/2} \frac{\sin^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x} dx \qquad ...(i)$$

$$I = \int_0^{\pi/2} \frac{\left[\sin\left(\frac{\pi}{2}-x\right)\right]^{3/2}}{\left[\sin\left(\frac{\pi}{2}-x\right)\right]^{3/2}+\left[\cos\left(\frac{\pi}{2}-x\right)\right]^{3/2}} dx$$

$$= \int_0^{\pi/2} \frac{\cos^{3/2} x}{\cos^{3/2} x + \sin^{3/2} x} dx \qquad ...(ii)$$

समीकरण (i) तथा (ii) को जोड़ने पर,

$$2I = \int_0^{\pi/2} \frac{\sin^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x} dx + \int_0^{\pi/2} \frac{\cos^{3/2} x}{\cos^{3/2} x + \sin^{3/2} x} dx$$

$$= \int_0^{\pi/2} \frac{\sin^{3/2} x + \cos^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x} dx$$

$$= \int_0^{\pi/2} 1 dx = [x]_0^{\pi/2} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$\therefore \qquad I = \frac{\pi}{4}.$ **उत्तर**

प्रश्न 4. $\int_0^{\pi/2} \frac{\cos^5 x}{\sin^5 x + \cos^5 x} dx$

हल : मान लीजिए

$$I = \int_0^{\pi/2} \frac{\cos^5 x}{\sin^5 x + \cos^5 x} dx \qquad ...(i)$$

$$I = \int_0^{\pi/2} \frac{\left[\cos\left(\frac{\pi}{2}-x\right)\right]^5}{\left[\sin\left(\frac{\pi}{2}-x\right)\right]^5+\left[\cos\left(\frac{\pi}{2}-x\right)\right]^5} dx$$

$$= \int_0^{\pi/2} \frac{\sin^5 x}{\cos^5 x + \sin^5 x} dx \qquad ...(ii)$$

समीकरण (i) तथा (ii) को जोड़ने पर,

$$2I = \int_0^{\pi/2} \frac{\cos^5 x}{\sin^5 x + \cos^5 x} dx + \int_0^{\pi/2} \frac{\sin^5 x}{\cos^5 x + \sin^5 x} dx$$

$$= \int_0^{\pi/2} \frac{\cos^5 x + \sin^5 x}{\sin^5 x + \cos^5 x} dx = \int_0^{\pi/2} 1 dx$$

$$= [x]_0^{\pi/2} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$\therefore \qquad I = \frac{\pi}{4}.$ **उत्तर**

प्रश्न 5. $\int_{-5}^{5} |x+2|\, dx$

हल :
$$\int_{-5}^{5} |x+2|\, dx = \int_{-5}^{-2} |x+2|\, dx + \int_{-2}^{5} |x+2|\, dx$$
$$= -\int_{-5}^{-2} |x+2|\, dx + \int_{-2}^{5} |x+2|\, dx$$
[∵ जब $x < -2, |x+2| = -(x+2)$
और जब $x > -2, |x+2| = x+2$]
$$= -\left[\frac{x^2}{2}+2x\right]_{-5}^{-2} + \left[\frac{x^2}{2}+2x\right]_{-2}^{5}$$
$$= -\left[\left(\frac{(-2)^2}{2}-4\right)-\left(\frac{25}{2}-10\right)\right]+\left[\left(\frac{25}{2}+10\right)-\left(\frac{4}{2}-4\right)\right]$$
$$= -\left[(-2)-\frac{5}{2}\right]+\left[\frac{45}{2}-(-2)\right] = \frac{9}{2}+\frac{49}{2} = \frac{58}{2}$$
$$= 29.$$
उत्तर

प्रश्न 6. $\int_{2}^{8} |x-5|\, dx$

हल :
$$\int_{2}^{8} |x-5|\, dx = \int_{2}^{5} |x-5|\, dx + \int_{5}^{8} |x-5|\, dx$$
$$\left[\because \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx\right]$$

जब $x < 5, |x-5| = -(x-5)$
और जब $x > 5, |x-5| = x-5$

$$\therefore \quad = -\left[\frac{x^2}{2}-5x\right]_{2}^{5} + \left[\frac{x^2}{2}-5x\right]_{5}^{8}$$
$$= -\left[\left(\frac{25}{2}-25\right)-\left(\frac{4}{2}-10\right)\right]+\left[\left(\frac{64}{2}-40\right)-\left(\frac{25}{2}-25\right)\right]$$
$$= -\left[-\frac{25}{2}-(-8)\right]+\left[-8-\left(-\frac{25}{2}\right)\right]$$
$$= -\left(\frac{-25+16}{2}\right)+\left(\frac{-16+25}{2}\right) = \frac{9}{2}+\frac{9}{2} = 9.$$
उत्तर

प्रश्न 7. $\int_{0}^{1} x(1-x)^n\, dx$

हल :
$$\int_{0}^{1} x(1-x)^n\, dx = \int_{0}^{1} (1-x)[1-(1-x)]^n\, dx$$
$$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$
$$= \int_{0}^{1} (1-x)x^n\, dx$$

$$= \int_0^1 (x^n - x^{n+1})\,dx = \left[\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right]_0^1$$

$$= \frac{1}{n+1} - \frac{1}{n+2} = \frac{n+2-n-1}{(n+1)(n+2)}$$

$$= \frac{1}{(n+1)(n+2)}.$$ उत्तर

प्रश्न 8. $\int_0^{\pi/4} \log(1 + \tan x)dx$.

हल : मान लीजिए $I = \int_0^{\pi/4} \log(1 + \tan x)dx$

$$I = \int_0^{\pi/4} \log\left[1 + \tan\left(\frac{\pi}{4} - x\right)\right]dx$$

$$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi/4} \log\left(1 + \frac{1-\tan x}{1+\tan x}\right)dx$$

$$= \int_0^{\pi/4} \log\left(\frac{1+\tan x + 1 - \tan x}{1+\tan x}\right)dx$$

$$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan x}\right)dx$$

$$= \int_0^{\pi/4} [\log 2 - \log(1+\tan x)]dx$$

$$= \log 2\int_0^{\pi/4} dx - \int_0^{\pi/4} \log(1+\tan x)]dx$$

$$I = \log 2[x]_0^{\pi/4} - I = \log 2.\ \frac{\pi}{4} - I$$

$$2I = (\log 2)\frac{\pi}{4}$$

$\therefore$ $$I = \frac{\pi}{8}\log 2.$$ उत्तर

प्रश्न 9. $\int_0^2 x\sqrt{2-x}\,dx$

हल : $$\int_0^2 x\sqrt{2-x}\,dx = \int_0^2 (2-x)\sqrt{2-(2-x)}\,dx$$

$$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$= \int_0^2 (2-x)\sqrt{x}\,dx = \int_0^2 \left(2x^{1/2} - x^{3/2}\right)dx$$

$$= \left[2.\frac{2}{3}.x^{3/2} - \frac{2}{5}.x^{5/2}\right]_0^2$$

$$= \frac{4}{3}.2^{3/2} - \frac{2}{5}.2^{5/2} - 0$$

$$= \frac{4}{3}.2\sqrt{2} - \frac{2}{5}4\sqrt{2} = \frac{8\sqrt{2}}{3} - \frac{8\sqrt{2}}{5} = \frac{16\sqrt{2}}{15}.$$

उत्तर

प्रश्न 10. $\int_0^{\pi/2} (2\log\sin x - \log\sin 2x)\,dx$

हल : मान लीजिए
$$I = \int_0^{\pi/2} (2\log\sin x - \log\sin 2x)\,dx \quad ...(i)$$

$\therefore$
$$I = \int_0^{\pi/2} [2\log\sin\left(\frac{\pi}{2} - x\right) - \log\sin 2\left(\frac{\pi}{2} - x\right)]dx$$

$$I = \int_0^{\pi/2} (2\log\cos x - \log\sin 2x)dx \quad ...(ii)$$

समीकरण (i) तथा (ii) को जोड़ने पर,

$$2I = \int_0^{\pi/2} \{[2(\log\sin x + \log\cos x)] - 2\log\sin 2x\}\,dx$$

$$= \int_0^{\pi/2} 2[\log\sin x\cos x - \log\sin 2x]\,dx$$

$$I = \int_0^{\pi/2} \left[\log\left(\frac{\sin 2x}{2}\right) - \log\sin 2x\right]dx$$

$$= \int_0^{\pi/2} (\log\sin 2x - \log 2 - \log\sin 2x)dx$$

$$= -\int_0^{\pi/2} \log 2dx$$

$$= -\log 2.[x]_0^{\pi/2} = -\frac{\pi}{2}\log 2 = \frac{\pi}{2}\log\frac{1}{2}.$$

उत्तर

प्रश्न 11. $\int_{-\pi/2}^{\pi/2} \sin^2 x\,dx$

हल : मान लीजिए
$$I = \int_{-\pi/2}^{\pi/2} \sin^2 x\,dx$$

यदि $f(-x) = f(x)$, तब $\int_{-a}^{a} f(x)\sin = 2\int_{-a}^{a} f(x)\,dx$

यहाँ पर $\sin^2(-x) = \sin^2 x$ [$\because$ $\sin^2 x$ एक सम फलन है]

$\therefore$
$$I = 2\int_0^{\pi/2} \sin^2 x\,dx \quad ...(i)$$

$$= 2\int_0^{\pi/2} \sin^2\left(\frac{\pi}{2} - x\right)dx \quad \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$= 2\int_0^{\pi/2} \cos^2 x\,dx \quad ...(ii)$$

समीकरण (i) तथा (ii) को जोड़ने पर,

$$2I = 2\int_0^{\pi/2} \sin^2 x\,dx + 2\int_0^{\pi/2} \cos^2 x\,dx$$

$$= 2\int_0^{\pi/2} (\sin^2 x + \cos^2 x) dx$$

$$= 2\int_0^{\pi/2} 1 dx = 2[x]_0^{\pi/2}$$

$$= 2.\frac{\pi}{2} = \pi$$

$\therefore$ $$I = \frac{\pi}{2}.$$ उत्तर

प्रश्न 12. $\int_0^{\pi} \frac{x\,dx}{1+\sin x}$

हल : मान लीजिए $$I = \int_0^{\pi} \frac{x\,dx}{1+\sin x} \quad ...(i)$$

$$= \int_0^{\pi} \frac{\pi - x}{1+\sin(\pi - x)} \quad ...(ii)$$

$$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

समीकरण (i) तथा (ii) को जोड़ने पर,

$$2I = \int_0^{\pi} \frac{x}{1+\sin x} dx + \int_0^{\pi} \frac{\pi - x}{1+\sin x} dx$$

$$= \int_0^{\pi} \frac{x + \pi - x}{1+\sin x} dx = \pi \int_0^{\pi} \frac{dx}{1+\sin x}$$

$$= \pi \int_0^{\pi} \frac{1-\sin x}{1-\sin^2 x} dx = \pi \int_0^{\pi} \frac{1-\sin x}{\cos^2 x} dx$$

$$= \pi \int_0^{\pi} (\sec^2 x - \sec x \tan x) dx$$

$$= \pi \int_0^{\pi} \sec^2 x\, dx - \pi \int_0^{\pi} \sec x \tan x\, dx = \pi \quad [\tan x - \sec x]_0^{\pi}$$

$$= \pi[0 - \sec\pi + \sec 0] = \pi[1+1] = 2\pi$$

$\therefore$ $$I = \pi.$$ उत्तर

प्रश्न 13. $\int_{\pi/2}^{\pi/2} \sin^7 x\, dx$.

हल : $\int_{\pi/2}^{\pi/2} \sin^7 x\, dx$

यहाँ $$f(x) = \sin^7 x$$

$\Rightarrow$ $$f(-x) = \sin^7(-x) = -\sin^7 x = -f(x)$$

अर्थात् f एक विषम फलन है।

$\therefore$ $$I = 0$$

$\therefore$ $$\int_{-\pi/2}^{\pi/2} \sin^7 x\, dx = 0.$$ उत्तर

प्रश्न 14. $\int_0^{2\pi} \cos^5 x\, dx$

हल : माना कि $$I = \int_0^{2\pi} \cos^5 x\, dx$$

यदि $f(x) = \cos^5 x, f(2\pi - x) = \cos^5 (2\pi - x)$
$= \cos^5 x = f(x)$

तो $I = \int_0^{2\pi} f(x) = 2\int_0^{a} f(x)\,dx$

$\therefore$ $I = \int_0^{2\pi} \cos^5 x\,dx = 2\int_0^{\pi} \cos^5 x\,dx$

$\int_0^{2a} f(x)\,dx = 0$ यदि $f(2a - x) = -f(x)$

$f(x) = \cos^5 x$

$\therefore$ $f(\pi - x) = \cos^5 (\pi - x) = (-\cos x)^5$
$= -\cos^5 x = -f(x)$

या $f(2a - x) = -f(x)$ $\left[\because \int_0^{2a} f(x)\,dx = 0\right]$

$\therefore$ $I = 2\int_0^{\pi} \cos^5 x\,dx = 0.$ **उत्तर**

प्रश्न 15. $\int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x}\,dx$.

हल : मान लीजिए $I = \int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x}\,dx$ ...(i)

$$= \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - x\right) - \cos\left(\frac{\pi}{2} - x\right)}{1 + \sin\left(\frac{\pi}{2} - x\right)\cos\left(\frac{\pi}{2} - x\right)}$$

$\left[\because \int_0^{a} f(x)\,dx = \int_0^{a} (a - x)dx\right]$

$\therefore$ $I = \int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \cos x \sin x}\,dx$ ...(ii)

समी. (i) और (ii) को जोड़ने पर

$$2I = \int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x}\,dx + \int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \cos x \sin x}\,dx$$

$$= \int_0^{\pi/2} \frac{\sin x - \cos x + \cos x - \sin x}{1 + \cos x \cos x}\,dx = 0$$

$\therefore$ $I = 0.$ **उत्तर**

प्रश्न 16. $\int_0^{\pi} \log (1 + \cos x)dx$.

हल : मान लीजिए $I = \int_0^{\pi} \log(1 + \cos x)dx$ ...(i)

$\therefore$ $I = \int_0^{\pi} \log[1 + \cos(\pi - x)]\,dx$

$\left[\because \int_0^{a} f(x)\,dx = \int_0^{a} f(a - x)dx\right]$

$= \int_0^{\pi} \log(1 - \cos x)\,dx$ ...(ii)

समी. (i) और (ii) को जोड़ने पर

$$2I = \int_0^{\pi} \log(1 + \cos x)\,dx + \int_0^{\pi} \log(1 - \cos x)dx$$

$$= \int_0^{\pi} [\log(1 + \cos x) + \log(1 - \cos x)]dx$$

$$= \int_0^{\pi} \log(1 + \cos x)(1 - \cos x)dx$$

$$= \int_0^{\pi} \log(1 - \cos^2 x)dx = \int_0^{\pi} \log\sin^2 x\,dx$$

$$= 2\int_0^{\pi} \log\sin x\,dx$$

$\therefore$ $$I = \int_0^{\pi} \log\sin x\,dx \qquad ...(iii)$$

$\therefore$ $$I = 2\int_0^{\pi/2} \log\sin x\,dx = 2I_1 \text{ (माना)} \qquad ...(iv)$$

जहाँ $$I_1 = \int_0^{\pi/2} \log\sin x\,dx \qquad ...(v)$$

$$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a - x)dx\right]$$

$$I_1 = \int_0^{\pi/2} \log\sin\left(\frac{\pi}{2} - x\right)dx$$

$$= \int_0^{\pi/2} \log\cos x\,dx \qquad ...(vi)$$

समी. (v) और (vi) को जोड़ने पर

$$2I_1 = \int_0^{\pi/2} \log\sin x\,dx + \int_0^{\pi/2} \log\cos x\,dx$$

$$= \int_0^{\pi/2} (\log\sin x + \log\cos x)dx$$

$$= \int_0^{\pi/2} \log\sin x\cos x\,dx$$

$$= \int_0^{\pi/2} \log\left(\frac{2\sin x\cos x}{2}\right)dx$$

$$= \int_0^{\pi/2} (\log\sin 2x - \log 2)\,dx$$

$$= \int_0^{\pi/2} \log\sin 2x\,dx - (\log 2)\int_0^{\pi/2} 1\,dx$$

$$= \int_0^{\pi/2} \log\sin 2x\,dx - \frac{\pi}{2}\log 2$$

मान लीजिए $2x = t$, हो, तब $2dx = dt$,

अत: $$2I_1 = \frac{1}{2}\int_0^{\pi} \log\sin t\,dt - \frac{\pi}{2}\log 2$$

$$= \frac{1}{2}\int_0^{\pi} \log\sin x\,dx - \frac{\pi}{2}\log 2$$

$$= \frac{1}{2} \times 2\int_0^{\pi/2} \log \sin x \, dx - \frac{\pi}{2} \log 2$$

$\therefore \quad 2I_1 = I_1 - \frac{\pi}{2} \log 2$ [समीकरण (v) से]

$$I_1 = -\frac{\pi}{2} \log 2$$

इसका मान समीकरण (iv) में रखने पर

$$I = 2I_1 = -\pi \log 2.$$ **उत्तर**

प्रश्न 17. $\int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}} dx$.

हल : मान लीजिए $\quad I = \int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}} dx$ ...(i)

$\therefore \quad I = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x} + \sqrt{a-(a-x)}} dx$

$$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$= \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x} + \sqrt{x}} dx$ ...(ii)

समी. (i) और (ii) को जोड़ने पर

$$2I = \int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}} dx + \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x} + \sqrt{x}} dx$$

$$= \int_0^a \frac{\sqrt{x} + \sqrt{a-x}}{\sqrt{x} + \sqrt{a-x}} dx = \int_0^a 1 dx = [x]_0^a = a$$

$\therefore \quad I = \frac{a}{2}.$ **उत्तर**

प्रश्न 18. $\int_0^4 |x-1| dx$.

हल : $\int_0^4 |x-1| dx = -\int_0^1 |x-1| dx + \int_1^4 |x-1| dx$

$$\left[\because \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx\right]$$

$$I = -\int_0^1 (x-1) dx + \int_1^4 (x-1) dx$$

[$\because |x-1| = -(x-1)$ यदि $x < 1$
तथा $|x-1| = x-1$ यदि $x > 1$]

$$= -\left[\frac{x^2}{2} - x\right]_0^1 + \left[\frac{x^2}{2} - x\right]_1^4$$

$$= -\left[\frac{1}{2}-1\right]+\left(\frac{16}{2}-4\right)-\left(\frac{1}{2}-1\right)$$

$$= \frac{1}{2}+4+\frac{1}{2}=5.$$ **उत्तर**

प्रश्न 19. दर्शाइए कि $\int_0^a f(x)g(x)dx = 2\int_0^a f(x)\,dx$, यदि f और g को $f(x) = f(a - x)$ एवं $g(x) + g(a - x) = 4$ के रूप में परिभाषित किया गया है।

हल : $$\int_0^a f(x)g(x)\,dx = \int_0^a f(a-x)g(a-x)dx \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)dx\right]$$

$\because$ दिया है $f(a - x) = f(x)$ तथा $g(x) + g(a - x) = 4$

या $$g(a - x) = 4 - g(x)$$

$$= \int_0^a f(x)[4-g(x)]dx$$

$$= \int_0^a 4f(x)dx - \int_0^a f(x)g(x)\,dx$$

$$= 4\int_0^a f(x)\,dx - \int_0^a f(x)g(x)\,dx$$

$\therefore$ $$2\int_0^a f(x)g(x)dx = 4\int_0^a f(x)\,dx$$

या $$\int_0^a f(x)g(x)dx = 2\int_0^a f(x)\,dx$$ **इति सिद्धम्।**

प्रश्न 20 एवं 21 में सही उत्तर का चयन कीजिए।

प्रश्न 20. $\int_{-\pi/2}^{\pi/2}(x^3 + x\cos x + \tan^5 x + 1)dx$ का मान है :

(A) 0 (B) 2 (C) π (D) 1

हल : मान लीजिए $$I = \int_{-\pi/2}^{\pi/2}(x^3 + x\cos x + \tan^5 x + 1)dx$$

$$= \int_{-\pi/2}^{\pi/2}(x^3 + x\cos x + \tan^5 x)dx + \int_{-\pi/2}^{\pi/2} 1dx$$

पुन: माना $$f(x) = x^3 + x\cos x + \tan^5 x$$

$\therefore$ $$f(-x) = (-x)^3 + (-x)\cos(-x) + \tan^5(-x)$$

$$= -x^3 - x\cos x - \tan^5 x$$

$$= -(x^3 + x\cos x + \tan^5 x)$$

$$= -f(x)$$

$\therefore$ $$I = \int_{-\pi/2}^{\pi/2} f(x) - \int_{-\pi/2}^{\pi/2} f(x) + [x]_{-\pi/2}^{\pi/2}$$

$$= 0+\left[\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right]$$

$\left[\because \int_{-a}^{a} f(x)dx = 0, \text{ यदि } f \text{ एक विषम फलन है}\right]$

$$= \frac{\pi}{2}+\frac{\pi}{2} = \frac{2\pi}{2} = \pi$$

अत: विकल्प (C) सही है। **उत्तर**

प्रश्न 21. $\int_a^{\pi/2} \log\left(\frac{4+3\sin x}{4+3\cos x}\right)dx$ **का मान है :**

(A) 2 (B) $\frac{3}{4}$ (C) 0 (D) – 2

हल : मान लीजिए

$$I = \int_0^{\pi/2} \log\left(\frac{4+3\sin x)}{4+3\cos x)}\right)dx \quad ...(i)$$

$$I = \int_0^{\pi/2} \log\left(\frac{4+3\sin\left(\frac{\pi}{2}-x\right)}{4+3\cos\left(\frac{\pi}{2}-x\right)}\right)dx$$

$$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)dx\right]$$

$$I = \int_0^{\pi/2} \log\left(\frac{4+3\cos x}{4+3\sin x}\right)dx \quad ..(ii)$$

समीकरण (i) तथा (ii) को जोड़ने पर,

$$2I = \int_0^{\pi/2} \log\left(\frac{4+3\sin x}{4+3\cos x}\right) + \log\left(\frac{4+3\cos x}{4+3\sin x}\right)dx$$

$$= \int_0^{\pi/2} \log\left(\frac{4+3\sin x}{4+3\cos x} \times \frac{4+3\cos x}{4+3\sin x}\right)dx$$

$$= \int_0^{\pi/2} \log 1dx = 0$$

या $I = 0$

अत: विकल्प (C) सही है। **उत्तर**

अध्याय 7 पर विविध प्रश्नावली

प्रश्न 1 से 24 तक के प्रश्नों के फलनों का समाकलन कीजिए—

प्रश्न 1. $\frac{1}{x-x^3}$.

हल :

$$\frac{1}{x-x^3} = \frac{1}{x(1-x^2)} = \frac{1}{x(1+x)(1-x)}$$

अब

$$\frac{1}{x(1+x)(1-x)} = \frac{A}{x} + \frac{B}{1+x} + \frac{C}{1-x}$$

∴

$$1 = A(1-x^2) + Bx(1-x) + Cx(1+x)$$

$x = 0$ रखने पर,

$$1 = A \text{ या } A = 1$$

$x = -1$ रखने पर,

$$1 = B(-1)(1+1) = -2B \text{ या } B = -\frac{1}{2}$$

$x = 1$ रखने पर,

$$1 = C.\ 1.\ (1 + 1) = 2C \text{ या } C = \frac{1}{2}$$

$$\therefore \quad \frac{1}{x - x^3} = \frac{1}{x} - \frac{1}{2(1+x)} + \frac{1}{2(1-x)}$$

अत:

$$\int \frac{1}{x - x^3} dx = \int \frac{1}{x} dx - \frac{1}{2} \int \frac{1}{1+x} dx + \frac{1}{2} \int \frac{1}{1-x} dx$$

$$= \log|x| - \frac{1}{2} \log|1+x| - \frac{1}{2} \log|1-x| + C$$

$$= \frac{1}{2} \log|x|^2 - \frac{1}{2} \log|1 - x^2| + C$$

$$= \frac{1}{2} \log\left|\frac{x^2}{1-x^2}\right| + C.$$

उत्तर

प्रश्न 2. $\dfrac{1}{\sqrt{x+a} + \sqrt{x+b}}$.

हल :

$$\int \frac{1}{\sqrt{x+u} + \sqrt{x+b}} dx = \int \frac{1}{\sqrt{x+a} + \sqrt{x+b}} \times \left(\frac{\sqrt{x+a} - \sqrt{x+b}}{\sqrt{x+a} - \sqrt{x+b}} \right) dx$$

$$= \int \frac{\sqrt{x+a} - \sqrt{x+b}}{x + a - x - b} dx$$

$$= \frac{1}{a-b} \int \left[(x+a)^{1/2} - (x+b)^{1/2} \right] dx$$

$$= \frac{1}{a-b} \left[\frac{(x+a)^{3/2}}{\frac{3}{2}} - \frac{(x+b)^{3/2}}{\frac{3}{2}} \right] + C$$

$$= \frac{2}{3(a-b)} \left[(x + a^{3/2} - (x+b)^{3/2} \right] + C.$$

उत्तर

प्रश्न 3. $\dfrac{1}{x\sqrt{ax - x^2}}$.

हल : $\displaystyle\int \frac{1}{x\sqrt{ax - x^2}} dx$

$x = \dfrac{a}{t}$ रखने पर तब $dx = -\dfrac{a}{t^2} dt$

$$= \int \frac{-\frac{a}{t^2}}{\frac{a}{t}\sqrt{\frac{a^2}{t} - \frac{a^2}{t^2}}} dt = -\frac{1}{a} \int \frac{\frac{1}{t}}{\frac{\sqrt{t-1}}{t}} dt$$

$$= -\frac{1}{a} \int \frac{1}{\sqrt{t-1}} dt = -\frac{1}{a} \int (t-1)^{-1/2} dt$$

$$= -\frac{1}{a}\cdot\frac{(t-1)^{1/2}}{\frac{1}{2}}+C = -\frac{2}{a}\sqrt{t-1}+C$$

$$= -\frac{2}{a}\sqrt{\frac{a}{x}-1}+C = -\frac{2}{a}\sqrt{\frac{a-x}{x}}+C.$$ उत्तर

प्रश्न 4. $\int \frac{dx}{x^2(x^4+1)^{3/4}}$.

हल :

$$\int \frac{dx}{x^2(x^4+1)^{3/4}}$$

$$= \int \frac{dx}{x^2(x^4)^{3/4}\left(1+\frac{1}{x^4}\right)^{3/4}}$$

$$= \int \frac{dx}{x^2 . x^3\left(1+\frac{1}{x^4}\right)^{3/4}}$$

$$= \int \left(1+\frac{1}{x^4}\right)^{-3/4} . x^{-5} dx$$

मान लीजिए $1+\frac{1}{x^4} = t$

या $1+x^{-4} = t$

तब $-4x^{-5}\,dx = dt$

$\Rightarrow$ $x^{-5}dx = -\frac{1}{4}dt$

$$= -\frac{1}{4}\int t^{-3/4}dt$$

$$= -\frac{1}{4}\cdot\frac{\left(1+\frac{1}{x^4}\right)^{1-\frac{3}{4}}}{1-\frac{3}{4}}+C$$

$$= -\left(1+\frac{1}{x^4}\right)^{1/4}+C.$$ उत्तर

प्रश्न 5. $\frac{1}{x^{1/2}+x^{1/3}}$.

हल : $\int \frac{1}{x^{1/2}+x^{1/3}}dx = \frac{1}{x^{1/3}(1+x^{1/6})}$

अब $x^{1/6} = t$ रखने पर या $x = t^6$ हो, तब $dx = 6t^5\,dt$

$$= \int \frac{6t^5}{t^3+t^2}dt = 6\int \frac{t^5}{t^2(t+1)}dt = 6\int \frac{t^3}{t+1}dt$$

$$= 6\int \frac{t^3+1-1}{t+1}dt = 6\int \left(\frac{t^3+1}{t+1} - \frac{1}{t+1}\right)dt$$

$$= 6\int \left(\frac{(t+1)(t^2-t+1)}{t+1} - \frac{1}{t+1}\right)dt$$

$$= 6\int \left(t^2 - t + 1 - \frac{1}{t+1}\right)dt$$

$$= 6\left(\frac{t^3}{3} - \frac{t^2}{2} + t - \log|t+1|\right) + C$$

$$= 6\left[\frac{1}{3}(x^{1/6})^3 - \frac{1}{2}(x^{1/6})^2 + x^{1/6} - \log(x^{1/6}+1)\right] + C$$

$$= 2\sqrt{x} - 3x^{1/3} + 6x^{1/6} - 6\log(x^{1/6}+1) + C.$$ **उत्तर**

प्रश्न 6. $\dfrac{5x}{(x+1)(x^2+9)}$.

हल : आंशिक भिन्न के प्रयोग से,

$$\frac{5x}{(x+1)(x^2+9)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+9}$$

$\therefore$
$$5x = A(x^2+9) + (Bx+C)(x+1)$$
$$= A(x^2+9) + B(x^2+x) + C(x+1)$$

$x = -1$ लेने पर,

$$-5 = A \times 10 \text{ या } A = -\frac{1}{2}$$

x^2 के गुणांकों की तुलना करने पर

$$0 = A + B \text{ या } B = -A \text{ या } B = \frac{1}{2}$$

अचर राशि की तुलना करने पर

$$0 = 9A + C \text{ या } C = -9A = \frac{9}{2}$$

$\therefore$
$$\frac{5x}{(x+1)(x^2+9)} = -\frac{1}{2(x+1)} + \frac{\frac{1}{2}x + \frac{9}{2}}{x^2+9}$$

$$= -\frac{1}{2(x+1)} + \frac{1}{2}\left(\frac{x+9}{x^2+9}\right)$$

$\therefore$
$$\int \frac{5x}{(x+1)(x^2+9)}dx = -\frac{1}{2}\int \frac{1}{x+1}dx + \frac{1}{2}\int \frac{x+9}{x^2+9}dx$$

$$= -\frac{1}{2}\log|x+1| + \frac{1}{4}\int \frac{2x}{x^2+9}dx + \frac{9}{2}\int \frac{1}{x^2+9}dx + C$$

$$= -\frac{1}{2}\log|x+1|+\frac{1}{4}\log|x^2+9|+\frac{9}{2}\times\frac{1}{3}\tan^{-1}\frac{x}{3}+C$$

$$= -\frac{1}{2}\log|x+1|+\frac{1}{4}\log|x^2+9|+\frac{3}{2}\tan^{-1}\frac{x}{3}+C.$$ उत्तर

प्रश्न 7. $\dfrac{\sin x}{\sin(x-a)}$.

हल : $\int\frac{\sin x}{\sin(x-a)}dx = \int\frac{\sin(x-a+a)}{\sin(x-a)}dx$

$$= \int\frac{\sin(x-a)\cos a+\cos(x-a)\sin a}{\sin(x-a)}dx$$

$$= \int[\cos a+\cot(x-a)\sin a]dx$$

$$= x\cos a + \sin a \log|\sin(x-a)| + C$$

$$= \sin a \log|\sin(x-a)| + x\cos a + C.$$

उत्तर

प्रश्न 8. $\dfrac{e^{5\log x}-e^{4\log x}}{e^{3\log x}-e^{2\log x}}$.

हल : $\int\frac{e^{5\log x}-e^{4\log x}}{e^{3\log x}-e^{2\log x}}dx$

$$= \int\frac{e^{\log x^5}-e^{\log x^4}}{e^{\log x^3}-e^{\log x^2}}dx$$

$$= \int\frac{x^5-x^4}{x^3-x^2}dx$$

$$= \int\frac{x^4(x-1)}{x^2(x-1)}dx$$

$$= \int x^2dx = \frac{x^3}{3}+C.$$ उत्तर

प्रश्न 9. $\dfrac{\cos x}{\sqrt{4-\sin^2 x}}$.

हल : $\int\frac{\cos x}{\sqrt{4-\sin^2 x}}dx$

मान लीजिए $\sin x = t$ हो, तब $\cos x\, dx = dt$

$$= \int\frac{dt}{\sqrt{4-t^2}} = \sin^{-1}\frac{t}{2}+C = \sin^{-1}\left(\frac{\sin x}{2}\right)+C.$$

उत्तर

प्रश्न 10. $\dfrac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}$

हल : $\int \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x} dx$

यहाँ $\sin^8 x - \cos^8 x$

$$= (\sin^4 x - \cos^4 x)(\sin^4 x + \cos^4 x)$$
$$= (\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x)(\sin^4 x + \cos^4 x)$$
$$= (\sin^2 x - \cos^2 x)\{(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x\}$$
$$[\because a^2 + b^2 = (a+b)^2 - 2ab]$$
$$= (\sin^2 - \cos^2 x)[1 - 2\sin^2 x \cos^2 x]$$

अत: $$\int \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x} dx = \int \frac{(\sin^2 x - \cos^2 x)(1 - 2\sin^2 x \cos^2 x)}{1 - 2\sin^2 x \cos^2 x} dx$$
$$= \int (\sin^2 x - \cos^2 x) dx = -\int \cos 2x \, dx$$
$$= -\frac{\sin 2x}{2} + C = -\frac{1}{2}\sin 2x + C.$$ उत्तर

प्रश्न 11. $\frac{1}{\cos(x+a)\cos(x+b)}$.

हल : $$\int \frac{1}{\cos(x+a)\cos(x+b)} dx = \frac{1}{\sin(a-b)} \int \frac{\sin(a-b)}{\cos(x+a)\cos(x+b)} dx$$
$$= \frac{1}{\sin(a-b)} \int \frac{\sin(x+a-x-b)}{\cos(x+a)\cos(x+b)} dx$$
$$= \frac{1}{\sin(a-b)} \int \frac{\sin(\overline{x+a} - \overline{x+b})}{\cos(x+a)\cos(x+b)} dx$$
$$= \frac{1}{\sin(a-b)} \int \frac{\sin(x+a)\cos(x+b) - \cos(x+a)\sin(x+b)}{\cos(x+a)\cos(x+b)} dx$$
$$= \frac{1}{\sin(a-b)} \int [(\tan(x+a) - \tan(x+b)] dx$$
$$= \frac{1}{\sin(a-b)} [-\log|\cos(x+a)| + \log|\cos(x+b)|] + C$$
$$= -\frac{1}{\sin(a-b)} \log\left|\frac{\cos(x+a)}{\cos(x+b)}\right| + C$$
$$= \frac{1}{\sin(a-b)} \log\left|\frac{\cos(x+b)}{\cos(x+a)}\right| + C.$$ उत्तर

प्रश्न 12. $\frac{x^3}{\sqrt{1-x^8}}$

हल : $$\int \frac{x^3}{\sqrt{1-x^8}} dx = \int \frac{x^3 dx}{\sqrt{1-(x^4)^2}}$$

मान लीजिए $x^4 = t$ रखने पर, $4x^3\, dx = dt$

$$= \frac{1}{4}\int \frac{dt}{\sqrt{1-t^2}} = \frac{1}{4}\sin^{-1} t + C$$

$$= \frac{1}{4}\sin^{-1} x^4 + C.$$ उत्तर

प्रश्न 13. $\dfrac{e^x}{(1+e^x)(2+e^x)}$.

हल : $\displaystyle\int \frac{e^x}{(1+e^x)(2+e^x)} dx$

यहाँ $e^x = t$ रखने पर, $e^x\, dx = dt$

$$= \int \frac{dt}{(1+t)(2+t)}$$

अब $\dfrac{1}{(1+t)(2+t)} = \dfrac{A}{1+t} + \dfrac{B}{2+t}$

$\therefore$ $1 = A(2+t) + B(1+t)$

$t = -1$ लेने पर, $1 = A \times 1$ या $A = 1$

तथा $t = -2$ लेने पर, $1 = B(1-2)$ या $B = -1$

$$= \int \frac{A}{1+t} dt + \int \frac{B}{2+t} dt$$

$$= \int \frac{1}{1+t} dt - \int \frac{1}{2+t} dt$$

$$= \log |1+t| - \log |2+t| + C$$

$$= \log\left(\frac{1+t}{2+t}\right) + C = \log\left|\frac{1+e^x}{2+e^x}\right| + C.$$ उत्तर

प्रश्न 14. $\dfrac{1}{(x^2+1)(x^2+4)}$

हल : $\displaystyle\int \frac{1}{(x^2+1)(x^2+4)}$

$x^2 = y$ रखने पर, $\dfrac{1}{(x^2+1)(x^2+4)} = \dfrac{1}{(y+1)(y+4)} = \dfrac{A}{y+1} + \dfrac{B}{y+4}$

$\therefore$ $1 = A(y+4) + B(y+1)$

$y = -1$ रखने पर, $1 = A(-1+4) = 3A$ या $A = \dfrac{1}{3}$

तथा $y = -4$ रखने पर, $1 = B(-4+1) = -3B$ या $B = -\dfrac{1}{3}$

अर्थात् $\dfrac{1}{(x^2+1)(x^2+4)} = \dfrac{1}{3(y+1)} - \dfrac{1}{3(y+4)} = \dfrac{1}{3(x^2+1)} - \dfrac{1}{3(x^2+4)}$

$$= \int \frac{1}{3(x^2+1)} dx - \frac{1}{3}\int \frac{1}{x^2+4} dx$$

$$= \frac{1}{3}\tan^{-1} x - \frac{1}{3}.\frac{1}{2}\tan^{-1}\frac{x}{2} + C$$

$$= \frac{1}{3}\tan^{-1} x - \frac{1}{6}\tan^{-1}\frac{x}{2} + C.$$ उत्तर

प्रश्न 15. $\cos^3 x e^{\log \sin x}$.

हल : $\int \cos^3 x e^{\log \sin x} dx = \int \cos^3 x \sin x\, dx$

मान लीजिए $\cos x = t$ हो, तब, $-\sin x\, dx = dt$ $[\because e^{\log x} = a]$

$$= \int t^3(-dt) = -\int t^3 dt = -\frac{t^4}{4} + C$$

$$= -\frac{\cos^4 x}{4} + C = -\frac{1}{4}\cos^4 x + C.$$ उत्तर

प्रश्न 16. $e^{3 \log x} (x^4 + 1)^{-1}$.

हल : $\int e^{3\log x}(x^4+1)^{-1} dx$

$$= \int \frac{e^{\log x^3}}{x^4+1} dx = \int \frac{x^3}{x^4+1} dx$$ $[\because e^{\log x} = x]$

मान लीजिए $x^4 + 1 = t$ हो, तब, $4x^3\, dx = dt$

$$= \frac{1}{4}\int \frac{dt}{t} = \frac{1}{4}\log|t| + C$$

$$= \frac{1}{4}\log|x^4+1| + C.$$ उत्तर

प्रश्न 17. $f'(ax + b)\, [f(ax + b)]^n$.

हल : $\int [f(ax+b)]^n f'(ax+b) dx$

अब $(ax + b) = t$ रखने पर

$\therefore$ $f'(ax + b).\, a dx = dt$

$$= \int t^n.\frac{1}{a} dt = \frac{1}{a}.\frac{t^{n-1}}{n+1} + C$$

$$= \frac{[f(ax+b)]^{n+1}}{a(n+1)} + C.$$ उत्तर

प्रश्न 18. $\dfrac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}}$.

हल : $\int \frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}} dx.$

अब $\sin^3 x \sin(x + \alpha) = \sin^3 x (\sin x \cos \alpha + \cos x \sin \alpha)$

$= \sin^4 x (\cos \alpha + \cot x \sin \alpha)$

$$= \int \frac{1}{\sqrt{\sin^4 x(\cos \alpha + \cot x \sin \alpha)}} dx$$

$$= \int \frac{1}{\sin^2 x\sqrt{\cos \alpha + \cot x \sin \alpha}} dx$$

$$= \int \frac{\operatorname{cosec}^2 x}{\sqrt{\cos \alpha + \cot x \sin \alpha}} dx$$

मान लीजिए $\cos \alpha + \cot x \sin \alpha = t$ हो, तब $\therefore -\operatorname{cosec}^2 x \sin \alpha \, dx = dt$

$$= -\frac{1}{\sin \alpha}\int \frac{dt}{\sqrt{t}} = -\frac{1}{\sin \alpha}\int t^{-1/2} dt$$

$$= -\frac{1}{\sin \alpha}\frac{t^{1/2}}{\frac{1}{2}} + C$$

$$= -\frac{2}{\sin \alpha}\sqrt{\cos \alpha + \cot x \sin \alpha} + C$$

$$= -\frac{2}{\sin \alpha}\sqrt{\cos \alpha + \frac{\cos x}{\sin x}\sin \alpha} + C$$

$$= -\frac{2}{\sin \alpha}\sqrt{\frac{\sin x \cos \alpha + \cos x \sin \alpha}{\sin x}} + C$$

$$= -\frac{2}{\sin \alpha}\sqrt{\frac{\sin (x+\alpha)}{\sin x}} + C.$$ उत्तर

प्रश्न 19. $\dfrac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}, (x \in [0, 1]).$

हल : $\displaystyle\int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}} dx$

चूँकि प्रतिलोम वृत्तीय फलन से,

$$^{-1}\sqrt{x} + \cos^{-1}\sqrt{x} = \frac{\pi}{2}$$

$$\therefore \quad \cos^{-1}\sqrt{x} = \frac{\pi}{2} - \sin^{-1}\sqrt{x}$$

$$= \int \frac{\sin^{-1}\sqrt{x} - \left(\frac{\pi}{2} - \sin^{-1}\sqrt{x}\right)}{\frac{\pi}{2}} dx$$

$$= \frac{2}{\pi}\int 2 \sin^{-1}\sqrt{x}\, dx - \int dx$$

$$= \frac{4}{\pi}\int \sin^{-1}\sqrt{x}\, dx - x$$

मान लीजिए $x = t^2$ हो, तब $dx = 2t\,dt$

$$= \frac{4}{\pi}\int \sin^{-1} t.2t\,dt - x$$

$$= -x + \frac{8}{\pi}\int (\sin^{-1} t)t\,dt$$

खण्डशः समाकलन करने पर,

$$= -x + \frac{8}{\pi}\left[(\sin^{-1} t)\frac{t^2}{2} - \int \frac{1}{\sqrt{1-t^2}}.\frac{t^2}{2}dt\right]$$

$$= -x + \frac{4}{\pi}t^2 \sin^{-1} t - \frac{4}{\pi}\int \frac{t^2}{\sqrt{1-t^2}}dt$$

$$= -x + \frac{4}{\pi}t^2 \sin^{-1} t + \frac{4}{\pi}\int \frac{1-t^2-1}{\sqrt{1-t^2}}dt$$

$$= -x + \frac{4}{\pi}t^2 \sin^{-1} t + \frac{4}{\pi}\int \left(\sqrt{1-t^2} - \frac{1}{\sqrt{1-t^2}}\right)dt$$

$$= -x + \frac{4}{\pi}t^2 \sin^{-1} t + \frac{4}{\pi}\left[\frac{t\sqrt{1-t^2}}{2} + \frac{1}{2}\sin^{-1} t - \sin^{-1} t\right] + C$$

$$= -x + \frac{4}{\pi}t^2 \sin^{-1} t + \frac{2}{\pi}t\sqrt{1-t^2} - \frac{2}{\pi}\sin^{-1} t + C$$

$$= -x + \frac{4}{\pi}x\sin^{-1}\sqrt{x} + \frac{2}{\pi}\sqrt{x}\sqrt{1-x} - \frac{2}{\pi}\sin^{-1}\sqrt{x} + C$$

$$= -x + \frac{2}{\pi}(2x-1)\sin^{-1}\sqrt{x} + \frac{2}{\pi}\sqrt{x}\,\sqrt{1-x} + C$$

$$= \frac{2(2x-1)}{\pi}\sin^{-1}\sqrt{x} + \frac{2\sqrt{x-x^2}}{\pi} - x + C.$$ उत्तर

प्रश्न 20. $\sqrt{\dfrac{1-\sqrt{x}}{1+\sqrt{x}}}$.

हल : $\int \sqrt{\dfrac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx$

मान लीजिए $\sqrt{x} = \cos t$ या $x = \cos^2 t$ रखने पर

$\therefore$ $dx = -2\cos t \sin t\,dt$

$$= \int \sqrt{\frac{1-\cos t}{1+\cos t}}(-2\cos t\,\sin t)dt$$

$$= -2\int \sqrt{\frac{2\sin^2 \frac{t}{2}}{2\cos^2 \frac{t}{2}}} \times \cos t\,\sin t\,dt$$

$$= -2\int \frac{\sin\frac{t}{2}}{\cos\frac{t}{2}}\left(2\sin\frac{t}{2}\cos\frac{t}{2}\cos t\right)dt$$

$$\left[\because \sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}\right]$$

$$= -4\int \sin^2\frac{t}{2}\cos t\,dt = -4\int\frac{1-\cos t}{2}\cos t\,dt$$

$$= -2\int(\cos t - \cos^2 t)\,dt$$

$$= -2\int\left[\cos t - \frac{1+\cos 2t}{2}\right]dt$$

$$= -2\sin t + \left(t + \frac{\sin 2t}{2}\right) + C$$

$$= t + \sin t\cos t - 2\sin t + C$$

अब पुन: $\cos t = \sqrt{x}$ रखने पर $\therefore \sin t = \sqrt{1-x}$, $t = \cos^{-1}\sqrt{x}$, के मान रखने पर

$$= \cos^{-1}\sqrt{x} - 2\sqrt{1-x} + \sqrt{x}\sqrt{1-x} + C$$

$$= -2\sqrt{1-x} + \cos^{-1}\sqrt{x} + \sqrt{x - x^2} + C.$$ उत्तर

प्रश्न 21. $\dfrac{2+\sin 2x}{1+\cos 2x}e^x.$

हल : $\displaystyle\int\frac{2+\sin 2x}{1+\cos 2x}e^x dx = \int\frac{2+2\sin x\cos x}{2\cos^2 x}e^x dx$

$$= \int\frac{1+\sin x\cos x}{\cos^2 x}e^x dx$$

$$= \int e^x(\tan x + \sec^2 x)dx$$

अब $e^x \tan x = t$ रखने पर

$(e^x\sec^2 x + e^x\tan x)\,dx = dt$ या $e^x(\tan x + \sec^2 x)\,dx = dt$

$$= \int dt = t + C = e^x\tan x + C.$$ उत्तर

प्रश्न 22. $\dfrac{x^2+x+1}{(x+1)^2(x+2)}$

हल : $\displaystyle\int\frac{x^2+x+1}{(x+1)^2(x+2)}dx$

अब $$\frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{A}{x+2} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$\therefore$ $$x^2+x+1 = A(x+1)^2 + B(x+2)(x+1) + C(x+2)$$
$$= A(x^2+2x+1) + B(x^2+3x+2) + C(x+2)$$

$x = -2$ लेने पर,

$$4 - 2 + 1 = A(-1)^2 \text{ या } 3 = A \text{ या } A = 3$$

तथा $x = -1$ लेने पर,

$$1 - 1 + 1 = C(-1 + 2) = C \text{ या } C = 1$$

x^2 के गुणांकों की तुलना करने पर,

$$1 = A + B \text{ या } B = 1 - A = 1 - 3 = -2$$

$$\therefore \quad \frac{x^2+x+1}{(x+2)(x+1)^2} = \frac{3}{x+2} - \frac{2}{x+1} + \frac{1}{(x+1)^2}$$

अब
$$\int \frac{x^2+x+1}{(x+1)^2(x+2)} dx = 3\int \frac{1}{x+2} dx - 2\int \frac{1}{x+1} dx + \int \frac{1}{(x+1)^2} dx$$

$$= 3 \log|x+2| - 2 \log|x+1| + \int (x+1)^{-2} dx$$

$$= 3 \log|x+2| - 2 \log|x+1| + \frac{(x+1)^{-1}}{-1} + C$$

$$= 3 \log|x+2| - 2 \log|x+1| - \frac{1}{x+1} + C$$

$$= -2 \log|x+1| - \frac{1}{x+1} + 3 \log|x+2| + C.$$

उत्तर

प्रश्न 23. $\tan^{-1}\sqrt{\dfrac{1-x}{1+x}}$.

हल : माना कि
$$I = \int \tan^{-1}\sqrt{\frac{1-x}{1+x}}\, dx$$

मान लीजिए $x = \cos\theta$ हो, तब, $dx = -\sin\theta\, d\theta$

$$= -\int \tan^{-1}\left(\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right)\sin\theta\, d\theta$$

$$= -\int \tan^{-1}\left(\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}\right)\sin\theta\, d\theta$$

$$= -\int \tan^{-1}\left(\tan\frac{\theta}{2}\right)\sin\theta\, d\theta$$

$$= -\int \frac{\theta}{2}\sin\theta\, d\theta = -\frac{1}{2}\int \theta\sin\theta\, d\theta$$

खण्डशः समाकलन करने पर,

$$= -\frac{1}{2}[\theta(-\cos\theta) - \int 1.(-\cos\theta)d\theta]$$

$$= \frac{1}{2}\theta\cos\theta - \frac{1}{2}\sin\theta + C$$

पुनः $x = \cos\theta$ रखने पर $\theta = \cos^{-1} x$, $\sin\theta = \sqrt{1-x^2}$

$$= \frac{1}{2}x\cos^{-1}x - \frac{1}{2}\sqrt{1-x^2} + C$$

$$= \frac{1}{2}(x\cos^{-1}x - \sqrt{1-x^2}) + C.$$ उत्तर

प्रश्न 24. $\dfrac{\sqrt{x^2+1}\,[\log(x^2+1) - 2\log x]}{x^4}$.

हल : $\displaystyle\int \frac{\sqrt{x^2+1}\,[\log(x^2+1) - 2\log x]}{x^4}dx$

$$= \int \frac{\sqrt{x^2+1}\,\log\left(\frac{x^2+1}{x^2}\right)}{x^4}dx$$

$$= \int \frac{x\left(\sqrt{1+\frac{1}{x^2}}\right)\log\left(1+\frac{1}{x^2}\right)dx}{x^4}$$

$$= \int \frac{\sqrt{1+\frac{1}{x^2}}\,\log\left(1+\frac{1}{x^2}\right)}{x^3}dx$$

मान लीजिए $1+\frac{1}{x^2} = t$ हो, तब $\frac{-2}{x^3}dx = dt$

$$= \int \frac{\sqrt{t}\log t}{-2}dt$$

$$= -\frac{1}{2}\int \sqrt{t}\,\log t\,dt$$

$$= -\frac{1}{2}\log t \times \frac{t^{3/2}}{\frac{3}{2}} - \int \frac{t^{3/2}}{\frac{3}{2}} \times \frac{1}{t}dt$$

$$= -\frac{1}{3}\left[t^{3/2}\log t - \int \sqrt{t}\,dt\right]$$

$$= -\frac{1}{3}\left[t^{3/2}\log t - \frac{t^{3/2}}{\frac{3}{2}}\right] + C$$

$$= -\frac{1}{3}t^{3/2}\left[\log t - \frac{2}{3}\right] + C$$

$$= -\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{3/2}\left[\log\left(1+\frac{1}{x^2}\right) - \frac{2}{3}\right] + C.$$ उत्तर

प्रश्न 25 से 33 तक के प्रश्नों में निश्चित समाकलनों का मान ज्ञात कीजिए।

प्रश्न 25. $\int_{\pi/2}^{\pi} e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx.$

हल : $\int_{\pi/2}^{\pi} e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$

$$= \int_{\pi/2}^{\pi} e^x\left(\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right)dx$$

$$= -\int_{\pi/2}^{\pi} e^x\left(\cot\frac{x}{2}-\frac{1}{2}\text{cosec}^2\frac{x}{2}\right)dx$$

मान लीजिए $e^x \cot\frac{x}{2} = t$ हो, तो $\left(e^x\cot\frac{x}{2}-\frac{1}{2}e^x\text{cosec}^2\frac{x}{2}\right)dx = dt$

या $\quad e^x\left(\cot\frac{x}{2}-\frac{1}{2}\text{cosec}^2\frac{x}{2}\right)dx = dt$

$$= -\int dt = -t + C = -e^x\cot\frac{x}{2}+C$$

$$\therefore \quad \int_{\pi/2}^{\pi} e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx = -\int_{\pi/2}^{\pi} dt = [-t]_{\pi/2}^{\pi} + C = \left[-e^x\cot\frac{x}{2}\right]_{\pi/2}^{\pi}$$

$$= -e^{\pi}\cot\frac{\pi}{2}+e^{\pi/2}\cot\frac{\pi}{4}$$

$$= e^{\pi}(-0)+e^{\pi/2}.1$$

$$= e^{\pi/2}.$$

उत्तर

प्रश्न 26. $\int_0^{\pi/4}\frac{\sin x\cos x}{\cos^4 x+\sin^4 x}dx.$

हल : $\int_0^{\pi/4}\frac{\sin x\cos x}{\cos^4 x+\sin^4 x}dx$

अंश व हर को $\cos^4 x$ से भाग देने पर

$$\int_0^{\pi/4}\frac{\tan x\sec^2 x\,dx}{1+\tan^4 x}$$

मान लीजिए $\tan 2x = t$ हो, तब

$$2\tan x\sec^2 x\,dx = dt$$

अब $x = 0$ हो, तो $t = 0$ और जब $x = \pi/4$ हो, तब $t = 1$.

$$= \frac{1}{2}\int_0^1\frac{dt}{1+t^2}$$

$$= \frac{1}{2}|\tan^{-1}t|_0^1 = \frac{1}{2}\tan^{-1}1 = \frac{\pi}{8}.$$

उत्तर

प्रश्न 27. $\int_0^{\pi/2} \frac{\cos^2 x\,dx}{\cos^2 x + 4\sin^2 x}$.

हल :

$$\int_0^{\pi/2} \frac{\cos^2 x\,dx}{\cos^2 x + 4\sin^2 x} = \int_0^{\pi/2} \frac{\cos^2 x}{\cos^2 x + 4 - 4\cos^2 x} dx$$

$$= \int_0^{\pi/2} \frac{\cos^2 x}{4 - 3\cos^2 x} dx$$

$$= \frac{1}{3}\int_0^{\pi/2} \frac{3\cos^2 x}{4 - 3\cos^2 x} dx$$

$$= \frac{-1}{3}\int_0^{\pi/2} \frac{4 - 3\cos^2 x - 4}{4 - 3\cos^2 x} dx$$

$$= \frac{-1}{3}\int_0^{\pi/2} \frac{4 - 3\cos^2 x}{4 - 3\cos^2 x} dx - \frac{4}{3}\int_0^{\pi/2} \frac{dx}{4 - 3\cos^2 x}$$

$$= -\frac{1}{3}[x]_0^{\pi/2} - \frac{4}{3}\int_0^{\pi/2} \frac{dx}{4 - 3\cos^2 x}$$

$$= -\frac{\pi}{6} + \frac{4}{3}\int_0^{\pi/2} \frac{\sec^2 x}{4\sec^2 x - 3} dx$$

[$\cos^2$ से अंश व हर से भाग करने पर]

$$= -\frac{\pi}{6} + \frac{4}{3}\int_0^{\pi/2} \frac{\sec^2 x}{4(1 + \tan^2 x) - 3} dx$$

$$= -\frac{\pi}{6} + \frac{4}{3}\int_0^{\pi/2} \frac{\sec^2 x}{1 + 4\tan^2 x} dx$$

पुन: $\tan x = t$ रखने पर, $\sec^2 x\,dx = dt$, जब $x = \frac{\pi}{2}$ तो $t = \tan\frac{\pi}{2} = \infty$

जब $x = 0$ तो $t = 0$

$$= -\frac{\pi}{6} + \frac{4}{3}\int_0^{\infty} \frac{1}{1+4t^2} dt = -\frac{\pi}{6} + \frac{4}{3}\cdot\frac{1}{2}[\tan^{-1} 2t]_0^{\infty}$$

$$= -\frac{\pi}{6} + \frac{2}{3}\cdot\frac{\pi}{2} = -\frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{6}.$$

उत्तर

प्रश्न 28. $\int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$.

हल : $\int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$

$$= \int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{1 - [1 - \sin 2x]}} dx$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{1 - [\sin x - \cos x]^2}} dx$$

मान लीजिए $\sin x - \cos x = t$ हो, तब, $(\cos x + \sin x)\, dx = dt$

जब $x = \frac{\pi}{3}$ तो $t = \sin\frac{\pi}{3} - \cos\frac{\pi}{3} = \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{\sqrt{3}-1}{2}$

तथा जब $x = \frac{\pi}{6}$ तो $t = \sin\frac{\pi}{6} - \cos\frac{\pi}{6} = \frac{1}{2} - \frac{\sqrt{3}}{2} = -\frac{\sqrt{3}-1}{2}$

$$\therefore \qquad I = \int_{\frac{\sqrt{3}-1}{2}}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}} = \left[\sin^{-1} t\right]_{\frac{\sqrt{3}-1}{3}}^{\frac{\sqrt{3}-1}{2}}$$

$$= -\left[\sin^{-1}\frac{1-\sqrt{3}}{2} - \sin^{-1}\frac{\sqrt{3}-1}{2}\right]$$

$$= -\sin^{-1}\frac{1-\sqrt{3}}{2} + \sin^{-1}\frac{\sqrt{3}-1}{2}$$

$[\because \sin^{-1}(-x) = -\sin^{-1} x]$

$$= 2\sin^{-1}\left[\frac{\sqrt{3}-1}{2}\right].$$

उत्तर

प्रश्न 29. $\int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}}$.

हल : $\int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}}$

$$= \int_0^1 \frac{1}{\sqrt{1+x}-\sqrt{x}} \times \frac{\sqrt{1+x}+\sqrt{x}}{\sqrt{1+x}+\sqrt{x}} dx$$

$$= \int_0^1 \frac{\sqrt{1+x}+\sqrt{x}}{1+x-x} dx = \int_0^1 (\sqrt{1+x}+\sqrt{x}) dx$$

$$= \frac{2}{3}\left[(1+x)^{3/2} + x^{3/2}\right]_0^1 = \frac{2}{3}[2^{3/2} + 1 - 1]$$

$$= \frac{2}{3}.2\sqrt{2} = \frac{4\sqrt{2}}{3}.$$

उत्तर

प्रश्न 30. $\int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16\sin 2x} dx$.

हल : $\int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16\sin 2x} dx$

मान लीजिएँ $\sin x - \cos x = t$ हो, तब $(\cos x + \sin x)dx = dt$

जब $x = \frac{\pi}{4}$ तो $t = 0$ तथा जब $x = 0$ तो $t = -1$

$(\sin x - \cos x)^2 = 1 - 2\sin x \cos x = 1 - \sin 2x$

$$\therefore \qquad \sin 2x = 1 - (\sin x - \cos x)^2 = 1 - t^2$$

$$= \int_{-1}^{0} \frac{dt}{9 + 16(1-t^2)} = \int_{-1}^{0} \frac{dt}{25 - 16t^2}$$

$$= \frac{1}{16}\int_{-1}^{0} \frac{dt}{\frac{25}{16} - t^2}$$

$$= \frac{1}{16} \cdot \frac{1}{2} \cdot \frac{4}{5}\left[\log\left(\frac{\frac{5}{4}+t}{\frac{5}{4}-t}\right)\right]_{-1}^{0}$$

$$= \frac{1}{40}[(\log 1 - (\log 1 - \log 9)]$$

$$= \frac{1}{40}\left(\log\frac{1}{9}\right) = \frac{1}{40}\log 9.$$ **उत्तर**

प्रश्न 31. $\int_0^{\pi/2} \sin 2x \tan^{-1}(\sin x)\, dx$

हल : $\int_0^{\pi/2} \sin 2x \tan^{-1}(\sin x)\, dx = 2\int_0^{\pi/2} \sin x \cos x \tan^{-1}(\sin x)\, dx$

मान लीजिए $\sin x = t$ हो, तब, $\cos x\, dx = dt$

जब $x = \frac{\pi}{2}$ तो $t = 1$, तथा जब $x = 0$ तो $t = 0$

$$= 2\int_0^1 t \tan^{-1} t\, dt$$

खण्डशः समाकलन करने पर,

$$= 2\left[\tan^{-1} t . \frac{t^2}{2}\right]_0^1 - 2\int_0^{-1} \frac{1}{1+t^2} \times \frac{t^2}{2} dt$$

$$= \frac{\pi}{4} - \int_0^{-1} \frac{1+t^2-1}{1+t^2} dt = \frac{\pi}{4} - \int_0^1 \left(1 - \frac{1}{1+t^2}\right) dt$$

$$= \frac{\pi}{4} - [t - \tan^{-1} t]_0^1 = \frac{\pi}{4} - \frac{1}{2}(1 - \tan^{-1} 1 - 0)$$

$$= \frac{\pi}{4} - \left(1 - \frac{\pi}{4}\right) = \frac{\pi}{2} - 1.$$ **उत्तर**

प्रश्न 32. $\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx.$

हल :

$$I = \int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx \quad \text{...(i)}$$

$\therefore$

$$I = \int_0^{\pi} \frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x) + \tan(\pi - x)} dx$$

$$\left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x) dx\right]$$

$$= \int_0^{\pi} \frac{-(\pi - x)\tan x}{\sec x - \tan x} dx$$

[$\because$ $\sec(\pi - x) = -\sec x$ एवं $\tan(\pi - x) = -\tan x$]

$$I = \int_0^{\pi} \frac{(\pi - x)\tan x}{\sec x + \tan x} dx \quad \text{...(ii)}$$

समीकरण (i) तथा (ii) को जोड़ने पर,

$$2I = \int_0^{\pi} \frac{[x + (\pi - x)]\tan x}{\sec x + \tan x} dx$$

$$= \pi\int_0^{\pi} \frac{\tan x}{\sec x + \tan x} dx$$

$$= \pi\int_0^{\pi} \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x} + \frac{\sin x}{\cos x}} dx$$

$$= \pi\int_0^{\pi} \frac{\sin x\, dx}{1 + \sin x}$$

$$= \pi\int_0^{\pi} \frac{1 + \sin x - 1}{1 + \sin x} dx = \pi\int_0^{\pi}\left(1 - \frac{1}{1 + \sin x}\right)dx$$

$$= \pi[x]_0^{\pi} - \pi\int_0^{\pi} \frac{1}{1 + \sin x} \times \frac{1 - \sin x}{1 - \sin x} dx$$

$$= \pi^2 - \pi\int_0^{\pi} \frac{1 - \sin x}{\cos^2 x} dx$$

$$= \pi^2 - \pi\int_0^{\pi} (\sec^2 x - \sec x \tan x) dx$$

$$= \pi^2 - \pi[\tan x - \sec x]_0^{\pi}$$

$$= \pi^2 - \pi\,[(\tan\pi - \sec\pi) - \tan 0 - \sec 0)$$

$$= \pi^2 - \pi\,[1 + 1] = \pi^2 - 2\pi$$

$$\therefore \quad I = \frac{\pi^2 - 2\pi}{2} = \frac{\pi}{2}[\pi - 2].$$ **उत्तर**

प्रश्न 33. $\int_1^4 (|x-1| + |x-2| + |x-3|)dx.$

हल : मान लीजिए

$$I = \int_1^4 (|x-1| + |x-2| + |x-3|)dx$$

$$= \int_1^4 |x-1|\,dx + \int_1^4 |x-2|\,dx + \int_1^4 |x-3|\,dx$$

$$= I_1 + I_2 + I_3 \text{ (माना)}$$

$$I_1 = \int_1^4 |x-1|\,dx = \int_1^4 (x-1)dx \quad [\because x > 1]$$

$$= \left|\frac{x^2}{2} - x\right|_1^4 = (8 - 4) - \left(\frac{1}{2} - 1\right) = 4 + \frac{1}{2} = 4\frac{1}{2} \quad \text{...(i)}$$

$$I_2 = \int_1^4 |x-2|\,dx = \int_1^2 |x-2|\,dx + \int_2^4 |x-2|\,dx$$

$$= -\int_1^2 (x-2)dx + \int_2^4 (x-2)dx$$

$\because$ जब $1 < x < 2$ तो $|x-2| = -(x-2)$
तथा जब $2 < x < 4$ तो $|x-2| = x-2$

$\therefore$
$$I_2 = \left[-\frac{x^2}{2} + 2x\right]_1^2 + \left[\frac{x^2}{2} - 2x\right]_2^4$$

$$= \left[\left(-\frac{4}{2}+4\right) - \left(-\frac{1}{2}+2\right)\right] + \left[\left(\frac{16}{2}-8\right) - \left(\frac{4}{2}-4\right)\right]$$

$$= \left[2 - \frac{3}{2} + 2\right] = 4 - \frac{3}{2} = \frac{5}{2} = 2\frac{1}{2} \qquad \text{...(ii)}$$

और
$$I_3 = \int_1^4 |x-3|\,dx = \int_1^3 |x-3|\,dx + \int_3^4 |x-3|\,dx$$

$\because$ जब $1 < x < 3$ तो $|x-3| = -(x-3)$
तथा जब $3 < x < 4$ तो $|x-3| = x-3$

$\therefore$
$$I_3 = \int_1^3 -(x-3)dx + \int_3^4 (x-3)dx$$

$$= \left[-\frac{x^2}{2} + 3x\right]_1^3 + \left[\frac{x^2}{2} - 3x\right]_3^4$$

$$= \left[\left(-\frac{9}{2}+9\right) - \left(-\frac{1}{2}+3\right)\right] + \left[\left(\frac{16}{2}-12\right) - \left(\frac{9}{2}-9\right)\right]$$

$$= \left(\frac{9}{2} - \frac{5}{2}\right) + \left(-4 + \frac{9}{2}\right) = \frac{5}{2} \qquad \text{...(iii)}$$

समीकरण (i), (ii) व (iii) को जोड़ने पर,

$$I_1 + I_2 + I_3$$

$$= \frac{9}{2} + \frac{5}{2} + \frac{5}{2} = \frac{19}{2}.$$ उत्तर

निम्नलिखित को सिद्ध कीजिए (प्रश्न 34 से 39 तक) :

प्रश्न 34. $\int_1^3 \frac{dx}{x^2(x+1)} = \frac{2}{3} + \log\frac{2}{3}$

हल : बायाँ पक्ष : $\int_1^3 \frac{dx}{x^2(x+1)}$

यहाँ
$$\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$\therefore$ $\qquad 1 = Ax(x+1) + B(x+1) + Cx^2$

$x = 0$ लेने पर, $\qquad 1 = B \times 1$ या $B = 1$

$x = -1$ लेने पर, $\qquad 1 = C(-1)^2$ या $C = 1$

x^2 के गुणांकों की तुलना करने पर

$$0 = A + C \text{ या } A = -C = -1$$

$$\therefore \quad \frac{1}{x^2(x+1)} = \frac{-1}{x} + \frac{1}{x^2} + \frac{1}{x+1}$$

$$\therefore \quad \int_1^3 \frac{1}{x^2(x+1)} dx = \int_1^3 -\frac{1}{x} dx + \int_1^3 \frac{1}{x^2} dx + \int_1^3 \frac{1}{x+1} dx$$

$$= [-\log|x|]_1^3 - \left[\frac{1}{x}\right]_1^3 + [\log|x+1|]_1^3$$

$$= (-\log 3 + \log 1) + \left(-\frac{1}{3} + \frac{1}{1}\right) + \log 4 - \log 2$$

$$= -\log 3 + \frac{2}{3} + 2\log 2 - \log 2$$

$$= \frac{2}{3} + \log\frac{2}{3} = \text{दायाँ पक्ष।}$$

प्रश्न 35. $\int_0^1 xe^x dx = 1.$

हल : बायाँ पक्ष $\int_0^1 xe^x dx$

x को पहला फलन मानकर खण्डशः समाकलन करने पर

$$= [xe^x]_0^1 - \int_0^1 1.e^x dx$$

$$= 1.e^1 - [e^x]_0^1 = e - (e-1) = 1 = \text{दायाँ पक्ष।}$$

प्रश्न 36. $\int_{-1}^1 x^{17} \cos^4 x\, dx = 0.$

हल : मान लीजिए $\quad I = \int_{-1}^1 x^{17} \cos^4 x\, dx$

$$f(x) = x^{17} \cos^4 x, f(-x) = (-x)^{17} \cos^4 (-x)\, dx$$
$$= -(x^{17} \cos^4 x)$$

$$\therefore \quad f(-x) = -f(x) \int_{-a}^a f(x) dx = 0 \text{ यदि } f \text{ विषम फलन हो}$$

$$\therefore \quad I = 0 \text{ अर्थात् } \int_{-1}^1 x^{17} \cos^4 x\, dx = 0.$$ **इति सिद्धम्।**

प्रश्न 37. $\int_0^{\pi/2} \sin^3 x\, dx = \frac{2}{3}.$

हल : मान लीजिए $\quad I = \int_0^{\pi/2} \sin^3 x\, dx$

$$\because \quad \sin 3x = 3\sin x - 4\sin^3 x$$

$$\therefore \quad \sin^3 x = \frac{3\sin x - \sin 3x}{4}$$

$$\therefore \quad I = \frac{1}{4}\int_0^{\pi/2} (3\sin x - \sin 3x) dx$$

$$= \frac{1}{4}\left[-3\cos x + \frac{\cos 3x}{3}\right]_0^{\pi/2}$$

$$= \frac{1}{4}\left[0-\left(-3+\frac{1}{3}\right)\right] = \frac{1}{4}\times\frac{8}{3} = \frac{2}{3}.$$ इति सिद्धम्।

प्रश्न 38. $\int_0^{\pi/4} 2\tan^3 x\,dx = 1 - \log 2$.

हल : बायाँ पक्ष: $\int_0^{\pi/4} 2\tan^3 x\,dx = 2\int_0^{\pi/4}\tan^3 x\,dx$

$$= 2\int_0^{\pi/4}\tan x.\tan^2 x\,dx$$

$$= 2\int_0^{\pi/4}\tan x(\sec^2 x - 1)dx$$

$$= 2\int_0^{\pi/4}\tan x\sec^2 x\,dx - 2\int_0^{\pi/4}\tan x\,dx = 2[I_1 - I_2]$$

यहाँ $\tan x = t$ तब $\sec^2 x\,dx = dt$

जब $x = 0$ हो तब $t = 0$ और जब $x = \frac{\pi}{4}$ हो, तब $t = 1$

$\therefore$ $$I_1 = \int_0^1 dt = \left[\frac{t^2}{2}\right]_0^1 = \frac{1}{2}$$

और $$I_2 = \int_0^{\pi/4}\tan x\,dx = -\left[\log\cos x\right]_0^{\pi/4}$$

$$= -\left[\log\cos\frac{\pi}{4} - \log\cos 0\right]$$

$$= -\left[\log\frac{1}{\sqrt{2}} - \log 1\right] = -\log\frac{1}{\sqrt{2}} = \frac{1}{2}\log 2$$

अब $$2(I_1 - I_2) = 2\left[\frac{1}{2} - \frac{1}{2}\log 2\right] = 1 - \log 2 = \text{दायाँ पक्ष।}$$

प्रश्न 39. $\int_0^1 \sin^{-1} x\,dx = \frac{\pi}{2} - 1.$

हल : बायाँ पक्ष : $\int_0^1 \sin^{-1} x.1dx$

$\sin^{-1} x$ को पहला फलन मानकर खण्डशः समाकलन करने पर

$$= \left[(\sin^{-1} x)x\right]_0^1 - \int_0^1 \frac{1}{\sqrt{1-x^2}}.x\,dx$$

$$= \left[\sin^{-1} x\right]_0^1 + \frac{1}{2}\int_0^1 \frac{-2x}{\sqrt{1-x^2}}dx$$

$$= \frac{\pi}{2} + \frac{1}{2}\times 2\left[\sqrt{1-x^2}\right]_0^1$$

$$= \frac{\pi}{2} - 1 = \text{दायाँ पक्ष।}$$

प्रश्न 40. योगफल की सीमा के रूप में $\int_0^1 e^{2-3x}dx$ का मान ज्ञात कीजिए।

हल : मान लीजिए $\quad I = \int_0^1 e^{2-3x}dx$

$\because \quad \int_a^b f(x)dx = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + ... + f(a+\overline{n-1}h)]$

जबकि $\quad h = \frac{1-0}{n} = \frac{1}{n}$ या $nh = 1 \qquad [\because a = 0, b = 1]$

$f(x) = e^{2-3x}, f(a) = e^{2-0} = e^2, f(a+h) = e^{2-3h}$

$f(a+2h) = e^{2-6h}, f(a+3h) = e^{2-9h}, ...$

$f(a+\overline{n-1}h) = e^{2-3(n-1)h}$

$\therefore \quad \int_0^1 e^{2-3x}dx = \lim_{h\to 0} h[e^2 + e^{2-3h} + e^{2-6h} + e^{2-9h} +....+ e^{2-3(n-1)h}]$

जबकि $\quad nh \to b - a = 1 - 0 = 1$

$\therefore \quad I = \lim_{h\to 0} he^2[1+e^{-3h}+e^{-6h}+e^{-9h}+....+e^{-3(n-1)h}]$

$= \lim_{h\to 0} he^2\left[\frac{[e^{-3h}]^n - 1}{e^{-3h}-1}\right] \quad \left[\because S = \frac{a(r^n-1)}{a-1} \text{ गुणोत्तर श्रेढ़ी के लिए}\right]$

$= \lim_{h\to 0} he^2(e^{-3nh}-1)\left(\frac{-3h}{e^{-3h}-1}\right)\times\frac{1}{-3h}$

अब $nh \to 1, \lim_{h\to 0}\frac{-3h}{e^{-3h}-1} = \log e = 1$

$\therefore \quad I = e^2(e^{-3}-1).1\left(-\frac{1}{3}\right)$

$= -\frac{1}{3}[e^{-1} - e^2] = \frac{1}{3}\left(e^2 - \frac{1}{e}\right).$ **उत्तर**

प्रश्न 41 से 44 तक के प्रश्नों में सही उत्तर का चयन कीजिए :

प्रश्न 41. $\int\frac{dx}{e^x+e^{-x}}$ **बराबर है :**

(A) $\tan^{-1}(e^x) + C$ (B) $\tan^{-1}(e^{-x}) + C$

(C) $\log(e^x - e^{-x}) + C$ (D) $\log(e^x + e^{-x}) + C$

हल : $\quad \int\frac{dx}{e^x+e^{-x}} = \int\frac{dx}{e^x+\frac{1}{e^x}} = \int\frac{e^x dx}{e^{2x}+1}$

मान लीजिए $e^x = t$ हो, तब $e^x\,dx = dt$

$= \int\frac{dt}{t^2+1} = \tan^{-1} t + C = \tan^{-1} e^x + C.$

अत: विकल्प (A) सही है। **उत्तर**

प्रश्न 42. $\int \frac{\cos 2x}{(\sin x + \cos x)^2} dx$ **बराबर है :**

(A) $\frac{-1}{\sin x + \cos x} + C$ (B) $\log|\sin x + \cos x| + C$

(C) $\log|\sin x - \cos x| + C$ (D) $\frac{1}{(\sin x + \cos x)^2}$

हल :
$$\int \frac{\cos 2x}{(\sin x + \cos x)^2} dx = \int \frac{\cos^2 x - \sin^2 x}{(\cos x + \sin x)^2} dx$$
$$= \int \frac{(\cos x + \sin x)(\cos x - \sin x)}{(\cos x + \sin x)^2} dx$$
$$= \int \frac{\cos x - \sin x}{\cos x + \sin x} dx$$

मान लीजिए $\cos x + \sin x = t$ हो, तब $-\sin x + \cos x\, dx = dt$
$$= \int \frac{dt}{t} = \log|t| + C = \log|\sin x + \cos x| + C$$

अत: विकल्प (B) सही है। **उत्तर**

प्रश्न 43. यदि $f(a + b - x) = f(x)$, तो $\int_a^b xf(x)dx$ **बराबर है :**

(A) $\frac{a+b}{2}\int_a^b f(b - x)dx$ (B) $\frac{a+b}{2}\int_a^b f(b + x)dx$

(C) $\frac{b-a}{2}\int_a^b f(x)dx$ (D) $\frac{a+b}{2}\int_a^b f(x)dx$

हल : मान लीजिए
$$I = \int_a^b xf(x)dx$$
$$= \int_a^b (a+b-x)f(a+b-x)dx$$
$$\left[\because \int_a^b f(x) = \int_a^b f(a+b-x)dx\right]$$
$$= \int_a^b f(a+b-x)f(x)dx$$

$\therefore$
$$I = \int_a^b [(a+b)f(x) - xf(x)]dx$$
$$= (a+b)\int_a^b f(x)dx - \int_a^b xf(x)dx$$
$$= (a+b)\int_a^b f(x)dx - I$$
$$2I = (a+b)\int_a^b f(x)dx$$

$\therefore$
$$I = \frac{a+b}{2}\int_a^b f(x)$$

अत: विकल्प (D) सही है। **उत्तर**

प्रश्न 44. $\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx$ **का मान है :**

(A) 1 (B) 0

(C) – 1 (D) $\frac{\pi}{4}$

हल : मान लीजिए

$$I = \int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx = \int_0^1 \tan^{-1}\left[\frac{x+(x-1)}{1-x(x-1)}\right]$$

$$= \int_0^1 \left[\tan^{-1} x + \tan^{-1}(x-1)\right]dx \quad ...(i)$$

$$= \int_0^1 [\tan^{-1}(1-x) + \tan^{-1}(1-x-1)]dx$$

$$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$= \int_0^1 \left[-\tan^{-1}(x-1) - \tan^{-1} x\right]dx$$

$$I = -\int_0^1 \tan^{-1} x + \tan^{-1}(x-1)dx \quad ...(ii)$$

समीकरण (i) तथा (ii) को जोड़ने पर,

$$2I = 0 \text{ या } I = 0$$

अतः विकल्प (B) सही है। **उत्तर**