Chapter 7 Integrals

EXERCISE 7.1

Question 1:

Find an anti-derivative (or integral) of the following functions by the method of inspection, $\sin 2x$.

Solution:

$$\Rightarrow \frac{d}{dx}(\cos 2x) = -2\sin 2x$$

$$\Rightarrow \sin 2x = -\frac{1}{2}\frac{d}{dx}(\cos 2x)$$

$$\Rightarrow \sin 2x = \frac{d}{dx}\left(-\frac{1}{2}\cos 2x\right)$$

Thus, the anti-derivative of $\sin 2x$ is $-\frac{1}{2}\cos 2x$

Question 2:

Find an anti-derivative (or integral) of the following functions by the method of inspection, $\cos 3x$.

Solution:

$$\Rightarrow \frac{d}{dx}(\sin 3x) = 3\cos 3x$$

$$\Rightarrow \cos 3x = \frac{1}{3}\frac{d}{dx}(\sin 3x)$$

$$\Rightarrow \cos 3x = \frac{d}{dx}\left(\frac{1}{3}\sin 3x\right)$$

Thus, the anti-derivative of $\cos 3x$ is $\frac{1}{3}\sin 3x$

Question 3:

Find an anti-derivative (or integral) of the following functions by the method of inspection, e^{2x}.

Solution:

$$\Rightarrow \frac{d}{dx}\left(e^{2x}\right) = 2e^{2x}$$

$$\Rightarrow e^{2x} = \frac{1}{2}\frac{d}{dx}\left(e^{2x}\right)$$

$$\Rightarrow e^{2x} = \frac{d}{dx}\left(\frac{1}{2}e^{2x}\right)$$

Thus, the anti-derivative of e^{2x} is $\frac{1}{2}e^{2x}$.

Question 4:

Find an anti-derivative (or integral) of the following functions by the method of inspection, $\left(ax+b\right)^2$.

Solution:

$$\frac{d}{dx}\left(ax+b\right)^3 = 3a\left(ax+b\right)^2$$

$$\Rightarrow \left(ax+b\right)^2 = \frac{1}{3a}\frac{d}{dx}\left(ax+b\right)^3$$

$$\Rightarrow \left(ax+b\right)^2 = \frac{d}{dx}\left(\frac{1}{3a}\left(ax+b\right)^3\right)$$

Thus, the anti-derivative $\left(ax+b\right)^2$ of is $\frac{1}{3a}\left(ax+b\right)^3$

Question 5:

Find an anti-derivative (or integral) of the following functions by the method of inspection, $\sin 2x - 4e^{3x}$

Solution:

$$\frac{d}{dx}\left(-\frac{1}{2}\cos 2x - \frac{4}{3}e^{3x}\right) = \sin 2x - 4e^{3x}$$

Thus, the anti-derivative of $\sin 2x - 4e^{3x}$ is $\left(-\frac{1}{2}\cos 2x - \frac{4}{3}e^{3x}\right)$

Find the following integrals in Exercises 6 to 20:

Question 6:

$$\int \left(4e^{3x} + 1\right)dx$$

Solution:

$$\int\left(4e^{3x}+1\right)dx = 4\int e^{3x}dx + \int 1\,dx$$
$$= 4\left(\frac{e^{3x}}{3}\right) + x + C$$
$$= \frac{4}{3}e^{3x} + x + C$$

Question 7:

$$\int x^2\left(1-\frac{1}{x^2}\right)dx$$

Solution:

$$\int x^2\left(1-\frac{1}{x^2}\right)dx = \int\left(x^2-1\right)dx$$
$$= \int x^2dx - \int 1\,dx$$
$$= \frac{x^3}{3} - x + C$$

Question 8:

$$\int\left(ax^2+bx+c\right)dx$$

Solution:

$$\int\left(ax^2+bx+c\right)dx = a\int x^2dx + b\int xdx + c\int 1dx$$
$$= a\left(\frac{x^3}{3}\right) + b\left(\frac{x^2}{2}\right) + cx + C$$
$$= \frac{ax^3}{3} + \frac{bx^2}{2} + cx + C$$

Question 9:

$$\int\left(2x^2+e^x\right)dx$$

Solution:

$$\int \left(2x^2 + e^x \right) dx = 2\int x^2 dx + \int e^x dx$$

$$= 2\left(\frac{x^3}{3} \right) + e^x + C$$

$$= \frac{2}{3}x^3 + e^x + C$$

Question 10:

$$\int \left(\sqrt{x} - \frac{1}{\sqrt{x}} \right)^2 dx$$

Solution:

$$\int \left(\sqrt{x} - \frac{1}{\sqrt{x}} \right)^2 dx = \int \left(x + \frac{1}{x} - 2 \right)$$

$$= \int x dx + \int \frac{1}{x} dx - 2\int 1 dx$$

$$= \frac{x^2}{2} + \log|x| - 2x + C$$

Question 11:

$$\int \frac{x^3 + 5x^2 - 4}{x^2} dx$$

Solution:

$$\int \frac{x^3 + 5x^2 - 4}{x^2} dx = \int \left(x + 5 - 4x^{-2} \right) dx$$

$$= \int x dx + 5\int 1 dx - 4\int x^{-2} dx$$

$$= \frac{x^2}{2} + 5x - 4\left(\frac{x^{-1}}{-1} \right) + C$$

$$= \frac{x^2}{2} + 5x + \frac{4}{x} + C$$

Question 12:

$$\int \frac{x^3 + 3x + 4}{\sqrt{x}} dx$$

Solution:

$$\int \frac{x^3 + 3x + 4}{\sqrt{x}} dx = \int \left(x^{\frac{5}{2}} + 3x^{\frac{1}{2}} + 4x^{-\frac{1}{2}} \right) dx$$

$$= \frac{x^{\frac{7}{2}}}{\frac{7}{2}} + \frac{3\left(x^{\frac{3}{2}}\right)}{\frac{3}{2}} + \frac{4\left(x^{\frac{1}{2}}\right)}{\frac{1}{2}} + C$$

$$= \frac{2}{7}x^{\frac{7}{2}} + 2x^{\frac{3}{2}} + 8x^{\frac{1}{2}} + C$$

$$= \frac{2}{7}x^{\frac{7}{2}} + 2x^{\frac{3}{2}} + 8\sqrt{x} + C$$

Question 13:

$$\int \frac{x^3 - x^2 + x - 1}{x - 1} dx$$

Solution:

$$\int \frac{x^3 - x^2 + x - 1}{x - 1} dx = \int \left[\frac{(x^2 + 1)(x - 1)}{x - 1} \right] dx$$

$$= \int (x^2 + 1) dx$$

$$= \int x^2 dx + \int 1 dx$$

$$= \frac{x^3}{3} + x + C$$

Question 14:

$$\int (1 - x)\sqrt{x}\, dx$$

Solution:

$$\int (1 - x)\sqrt{x}\, dx = \int \left(\sqrt{x} - x^{\frac{3}{2}} \right) dx$$

$$= \int x^{\frac{1}{2}} dx - \int x^{\frac{3}{2}} dx$$

$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + C$$

$$= \frac{2}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}} + C$$

Question 15:

$$\int \sqrt{x}\left(3x^2 + 2x + 3\right)dx$$

Solution:

$$\int \sqrt{x}\left(3x^2 + 2x + 3\right)dx = \int\left(3x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + 3x^{\frac{1}{2}}\right)dx$$

$$= 3\int x^{\frac{5}{2}}dx + 2\int x^{\frac{3}{2}}dx + 3\int x^{\frac{1}{2}}dx$$

$$= 3\left(\frac{x^{\frac{7}{2}}}{\frac{7}{2}}\right) + 2\left(\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right) + 3\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

$$= \frac{6}{7}x^{\frac{7}{2}} + \frac{4}{5}x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + C$$

Question 16:

$$\int \left(2x - 3\cos x + e^x\right)dx$$

Solution:

$$\int \left(2x - 3\cos x + e^x\right)dx = 2\int x\,dx - 3\int \cos x\,dx + \int e^x dx$$

$$= \frac{2x^2}{2} - 3(\sin x) + e^x + C$$

$$= x^2 - 3\sin x + e^x + C$$

Question 17:

$$\int \left(2x^2 - 3\sin x + 5\sqrt{x}\right)dx$$

Solution:

$$\int \left(2x^2 - 3\sin x + 5\sqrt{x}\right)dx = 2\int x^2 dx - 3\int \sin x\,dx + 5\int x^{\frac{1}{2}}dx$$

$$= \frac{2x^3}{3} - 3(-\cos x) + 5\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

$$= \frac{2}{3}x^3 + 3\cos x + \frac{10}{3}x^{\frac{3}{2}} + C$$

Question 18:

$\int \sec x (\sec x + \tan x) dx$

Solution:

$$\int \sec x (\sec x + \tan x) dx = \int \left(\sec^2 x + \sec x \tan x \right) dx$$
$$= \int \sec^2 x dx + \int \sec x \tan x dx$$
$$= \tan x + \sec x + C$$

Question 19:

$\int \dfrac{\sec^2 x}{\cos ec^2 x} dx$

Solution:

$$\int \frac{\sec^2 x}{\cos ec^2 x} dx = \int \frac{\dfrac{1}{\cos^2 x}}{\dfrac{1}{\sin^2 x}} dx$$
$$= \int \frac{\sin^2 x}{\cos^2 x} dx$$
$$= \int \tan^2 x dx$$
$$= \int \left(\sec^2 x - 1 \right) dx$$
$$= \int \sec^2 x dx - \int 1 dx$$
$$= \tan x - x + C$$

Question 20:

$$\int \frac{2-3\sin x}{\cos^2 x}dx$$

Solution:

$$\int \frac{2-3\sin x}{\cos^2 x}dx = \int \left(\frac{2}{\cos^2 x} - \frac{3\sin x}{\cos^2 x} \right)dx$$

$$= \int 2\sec^2 x\,dx - 3\int \tan x\sec x\,dx$$

$$= 2\tan x - 3\sec x + C$$

Choose the correct answer in Exercises 21 and 22

Question 21:

The anti-derivative of $\left(\sqrt{x} + \dfrac{1}{\sqrt{x}} \right)$ equals

(A) $\dfrac{1}{3}x^{\frac{1}{3}} + 2x^{\frac{1}{2}} + C$

(B) $\dfrac{2}{3}x^{\frac{2}{3}} + \dfrac{1}{2}x^2 + C$

(C) $\dfrac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$

(D) $\dfrac{3}{3}x^{\frac{3}{2}} + \dfrac{1}{2}x^{\frac{1}{2}} + C$

Solution:

$$\left(\sqrt{x} + \frac{1}{\sqrt{x}} \right) = \int x^{\frac{1}{2}}dx + \int x^{-\frac{1}{2}}dx$$

$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$$

Thus, the correct option is C.

Question 22:

If $\dfrac{d}{dx} f(x) = 4x^3 - \dfrac{3}{x^4}$ such that $f(2) = 0$, then $f(x)$ is

(A) $x^4 + \dfrac{1}{x^3} - \dfrac{129}{8}$

(B) $x^3 + \dfrac{1}{x^4} + \dfrac{129}{8}$

(C) $x^4 + \dfrac{1}{x^3} + \dfrac{129}{8}$

(D) $x^3 + \dfrac{1}{x^4} - \dfrac{129}{8}$

Solution:

Given, $\dfrac{d}{dx} f(x) = 4x^3 - \dfrac{3}{x^4}$

Anti-derivative of $4x^3 - \dfrac{3}{x^4} = f(x)$

Therefore,

$$f(x) = \int 4x^3 - \dfrac{3}{x^4} dx$$

$$f(x) = 4 \int x^3 dx - 3 \int (x^{-4}) dx$$

$$f(x) = 4 \left(\dfrac{x^4}{4} \right) - 3 \left(\dfrac{x^{-3}}{-3} \right) + C$$

$$f(x) = x^4 + \dfrac{1}{x^3} + C$$

Also,

$$\Rightarrow f(2) = 0$$

$$\Rightarrow f(2) = (2)^4 + \dfrac{1}{(2)^3} + C = 0$$

$$\Rightarrow 16 + \dfrac{1}{8} + C = 0$$

$$\Rightarrow C = - \left(16 + \dfrac{1}{8} \right)$$

$$\Rightarrow C = -\dfrac{129}{8}$$

$$\Rightarrow f(x) = x^4 + \dfrac{1}{x^3} - \dfrac{129}{8}$$

Thus, the correct option is A.

EXERCISE 7.2

Integrate the functions in Exercises 1 to 37:

Question 1:

$$\frac{2x}{1+x^2}$$

Solution:

Put $1+x^2 = t$

Therefore, $2x\,dx = dt$

$$\int \frac{2x}{1+x^2}\,dx = \int \frac{1}{t}\,dt = \log|t| + C$$
$$= \log|1+x^2| + C$$
$$= \log(1+x^2) + C$$

Question 2:

$$\frac{(\log x)^2}{x}$$

Solution:

Put $\log|x| = t$

Therefore, $\dfrac{1}{x}\,dx = dt$

$$\int \frac{(\log|x|)^2}{x}\,dx = \int t^2\,dt$$
$$= \frac{t^3}{3} + C$$
$$= \frac{(\log|x|)^3}{3} + C$$

Question 3:

$$\frac{1}{x + x\log x}$$

Solution:

$$\frac{1}{x + x\log x} = \frac{1}{x(1+\log x)}$$

Put $1 + \log x = t$

Therefore, $\dfrac{1}{x}dx = dt$

$$\int \dfrac{1}{x(1+\log x)}dx = \int \dfrac{1}{t}dt = \log|t| + C$$
$$= \log|1+\log x| + C$$

Question 4:
$\sin x \sin(\cos x)$

Solution:
Put $\cos x = t$
Therefore, $-\sin x\,dx = dt$
$$\int \sin x \sin(\cos x)\,dx = -\int \sin t\,dt = -[-\cos t] + C$$
$$= \cos t + C$$
$$= \cos(\cos x) + C$$

Question 5:
$\operatorname{Sin}(ax+b)\cos(ax+b)$

Solution:
$$\sin(ax+b)\cos(ax+b) = \dfrac{2\sin(ax+b)\cos(ax+b)}{2}$$
$$= \dfrac{\sin 2(ax+b)}{2}$$

Put $2(ax+b) = t$
Therefore, $2a\,dx = dt$
$$\int \dfrac{\sin 2(ax+b)}{2}dx = \dfrac{1}{2}\int \dfrac{\sin t\,dt}{2a}$$
$$= \dfrac{1}{4a}[-\cos t] + C$$
$$= \dfrac{-1}{4a}\cos 2(ax+b) + C$$

Question 6:
$\sqrt{ax+b}$
Solution:
Put $ax+b = t$
Therefore,

$$\Rightarrow adx = dt$$

$$\Rightarrow dx = \frac{1}{a}dt$$

$$\int (ax+b)^{\frac{1}{2}} dx = \frac{1}{a}\int t^{\frac{1}{2}}dt = \frac{1}{a}\left(\frac{t^{\frac{1}{2}}}{\frac{3}{2}}\right) + C$$

$$= \frac{2}{3a}(ax+b)^{\frac{3}{2}} + C$$

Question 7:

$$x\sqrt{x+2}$$

Solution:

Put, $x+2 = t$

$\therefore dx = dt$

$$\Rightarrow \int x\sqrt{x+2} = \int (t-2)\sqrt{t}\,dt$$

$$= \int \left(t^{\frac{3}{2}} - 2t^{\frac{1}{2}}\right)dt$$

$$= \int t^{\frac{3}{2}}dt - 2\int t^{\frac{1}{2}}dt$$

$$= \frac{t^{\frac{5}{2}}}{\frac{5}{2}} - 2\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

$$= \frac{2}{5}t^{\frac{5}{2}} - \frac{4}{3}t^{\frac{3}{2}} + C$$

$$= \frac{2}{5}(x+2)^{\frac{5}{2}} - \frac{4}{3}(x+2)^{\frac{3}{2}} + C$$

Question 8:

$$x\sqrt{1+2x^2}$$

Solution:

Put, $1 + 2x^2 = t$

$\therefore 4x\,dx = dt$

$$\Rightarrow \int x\sqrt{1+2x^2}\,dx = \int \frac{\sqrt{t}}{4}\,dt$$

$$= \frac{1}{4}\int t^{\frac{1}{2}}\,dt$$

$$= \frac{1}{4}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

$$= \frac{1}{6}\left(1+2x^2\right)^{\frac{3}{2}} + C$$

Question 9:

$$(4x+2)\sqrt{x^2+x+1}$$

Solution:

Put, $x^2 + x + 1 = t$

$\therefore (2x+1)\,dx = dt$

$$\int (4x+2)\sqrt{x^2+x+1}\,dx$$

$$= \int 2\sqrt{t}\,dt$$

$$= 2\int \sqrt{t}\,dt$$

$$= 2\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C = \frac{4}{3}\left(x^2+x+1\right)^{\frac{3}{2}} + C$$

Question 10:

$$\frac{1}{x-\sqrt{x}}$$

Solution:

$$\frac{1}{x - \sqrt{x}} = \frac{1}{\sqrt{x}\left(\sqrt{x} - 1\right)}$$

Put, $\left(\sqrt{x} - 1\right) = t$

$\therefore \frac{1}{2\sqrt{x}} dx = dt$

$\Rightarrow \int \frac{1}{\sqrt{x}\left(\sqrt{x} - 1\right)} dx = \int \frac{2}{t} dt$

$= 2 \log |t| + C$

$= 2 \log \left|\sqrt{x} - 1\right| + C$

Question 11:

$$\frac{x}{\sqrt{x + 4}}, x > 0$$

Solution:

Put, $x + 4 = t$

$\therefore dx = dt$

$\int \frac{x}{\sqrt{x + 4}} dx = \int \frac{(t - 4)}{\sqrt{t}} dt = \int \left(\sqrt{t} - \frac{4}{\sqrt{t}}\right) dt$

$= \left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) - 4\left(\frac{t^{\frac{1}{2}}}{\frac{1}{2}}\right) + C = \frac{2}{3}(t)^{\frac{3}{2}} - 8(t)^{\frac{1}{2}} + C$

$= \frac{2}{3} t \cdot t^{\frac{1}{2}} - 8t^{\frac{1}{2}} + C$

$= \frac{2}{3} t^{\frac{1}{2}} (t - 12) + C$

$= \frac{2}{3}(x + 4)^{\frac{1}{2}} (x + 4 - 12) + C$

$= \frac{2}{3} \sqrt{x + 4} (x - 8) + C$

Question 12:

$$\left(x^3 - 1\right)^{\frac{1}{3}} x^5$$

Solution:

Put, $x^3 - 1 = t$

$\therefore 3x^2 dx = dt$

$\Rightarrow \int \left(x^3 - 1\right)^{\frac{1}{3}} x^5 dx = \int \left(x^3 - 1\right)^{\frac{1}{3}} x^3 x^2 dx$

$\Rightarrow \int t^{\frac{1}{3}} (t+1) \frac{dt}{3} = \frac{1}{3} \int \left(t^{\frac{4}{3}} + t^{\frac{1}{3}}\right) dt$

$= \frac{1}{3} \left[\dfrac{t^{\frac{7}{3}}}{\frac{7}{3}} + \dfrac{t^{\frac{4}{3}}}{\frac{4}{3}} \right] + C$

$= \frac{1}{3} \left[\frac{3}{7} t^{\frac{7}{3}} + \frac{3}{4} t^{\frac{4}{3}} \right] + C$

$= \frac{1}{7} \left(x^3 - 1\right)^{\frac{7}{3}} + \frac{1}{4} \left(x^3 - 1\right)^{\frac{4}{3}} + C$

Question 13:

$$\frac{x^2}{\left(2 + 3x^3\right)^3}$$

Solution:

Put, $2 + 3x^3 = t$

$\therefore 9x^2 dx = dt$

$\Rightarrow \int \frac{x^2}{\left(2 + 3x^3\right)^3} dx = \frac{1}{9} \int \frac{dt}{(t)^3}$

$= \frac{1}{9} \left[\frac{t^{-2}}{-2} \right] + C$

$= -\frac{1}{18} \left(\frac{1}{t^2} \right) + C$

$= \frac{-1}{18\left(2 + 3x^3\right)^2} + C$

Question 14:

$$\frac{1}{x(\log x)^m}, x > 0$$

Solution:

Put, $\log x = t$

$$\therefore \frac{1}{x} dx = dt$$

$$\Rightarrow \int \frac{1}{x(\log x)^m} dx = \int \frac{dt}{(t)^m} = \left(\frac{t^{-m-1}}{1-m}\right) + C$$

$$= \frac{(\log x)^{1-m}}{(1-m)} + C$$

Question 15:

$$\frac{x}{9 - 4x^2}$$

Solution:

Put, $9 - 4x^2 = t$

$$\therefore -8x \, dx = dt$$

$$\Rightarrow \int \frac{x}{9 - 4x^2} dx = \frac{-1}{8} \int \frac{1}{t} dt$$

$$= \frac{-1}{8} \log|t| + C$$

$$= \frac{-1}{8} \log|9 - 4x^2| + C$$

Question 16:

$$e^{2x+3}$$

Solution:

Put, $2x + 3 = t$

$$\therefore 2dx = dt$$

$$\Rightarrow \int e^{2x+3} dx = \frac{1}{2} \int e^t dt$$

$$= \frac{1}{2} (e^t) + C$$

$$= \frac{1}{2} e^{(2x+3)} + C$$

Question 17:

$$\frac{x}{e^{x^2}}$$

Solution:

Put, $x^2 = t$

$\therefore 2x\,dx = dt$

$$\Rightarrow \int \frac{x}{e^{x^2}}\,dx = \frac{1}{2}\int \frac{1}{e^t}\,dt = \frac{1}{2}\int e^{-t}\,dt$$

$$= \frac{1}{2}\left(\frac{e^{-t}}{-1}\right) + C$$

$$= -\frac{1}{2}e^{-x^2} + C$$

$$= \frac{-1}{2e^{x^2}} + C$$

Question 18:

$$\frac{e^{\tan^{-1} x}}{1 + x^2}$$

Solution:

Put, $\tan^{-1} x = t$

$$\therefore \frac{1}{1 + x^2}\,dx = dt$$

$$\Rightarrow \int \frac{e^{\tan^{-1} x}}{1 + x^2}\,dx = \int e^t\,dt$$

$$= e^t + C$$

$$= e^{\tan^{-1} x} + C$$

Question 19:

$$\frac{e^{2x} - 1}{e^{2x} + 1}$$

Solution:

$$\frac{e^{2x} - 1}{e^{2x} + 1}$$

Dividing Nr and Dr by e^x, we get

$$\frac{\dfrac{e^{2x}-1}{e^x}}{\dfrac{e^{2x}+1}{e^x}} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Let $e^x + e^{-x} = t$

$\left(e^x - e^{-x}\right)dx = dt$

$$\Rightarrow \int \frac{e^{2x}-1}{e^{2x}+1}dx = \int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx$$

$$= \int \frac{dt}{t}$$

$$= \log|t| + C$$

$$= \log\left|e^x + e^{-x}\right| + C$$

Question 20:

$$\frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}$$

Solution:

Put, $e^{2x} + e^{-2x} = t$

$\left(2e^{2x} - 2e^{-2x}\right)dx = dt$

$\Rightarrow 2\left(e^{2x} - e^{-2x}\right)dx = dt$

$$\Rightarrow \int \left(\frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}\right)dx = \int \frac{dt}{2t}$$

$$= \frac{1}{2}\int \frac{1}{t}dt$$

$$= \frac{1}{2}\log|t| + C$$

$$= \frac{1}{2}\log\left|e^{2x} + e^{-2x}\right| + C$$

Question 21:

$$\tan^2(2x-3)$$

Solution:

$\tan^2(2x-3) = \sec^2(2x-3) - 1$

Put, $2x - 3 = t$

$$\therefore 2dx = dt$$

$$\Rightarrow \int \tan^2(2x-3)\,dx = \int \left[\sec^2(2x-3)-1\right]dx$$

$$= \frac{1}{2}\int \left(\sec^2 t\right)dt - \int 1\,dx = \frac{1}{2}\int \sec^2 t\,dt - \int 1\,dx$$

$$= \frac{1}{2}\tan t - x + C$$

$$= \frac{1}{2}\tan(2x-3) - x + C$$

Question 22:

$$\sec^2(7-4x)$$

Solution:

Put, $7-4x = t$

$$\therefore -4dx = dt$$

$$\therefore \int \sec^2(7-4x)\,dx = \frac{-1}{4}\int \sec^2 t\,dt$$

$$= \frac{-1}{4}\left(\tan t\right) + C$$

$$= \frac{-1}{4}\tan(7-4x) + C$$

Question 23:

$$\frac{\sin^{-1}x}{\sqrt{1-x^2}}$$

Solution:

Put, $\sin^{-1}x = t$

$$\frac{1}{\sqrt{1-x^2}}\,dx = dt$$

$$\Rightarrow \int \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx = \int t\,dt$$

$$= \frac{t^2}{2} + C = \frac{\left(\sin^{-1}x\right)^2}{2} + C$$

Question 24:

$$\frac{2\cos x - 3\sin x}{6\cos x + 4\sin x}$$

Solution:

$$\frac{2\cos x - 3\sin x}{6\cos x + 4\sin x} = \frac{2\cos x - 3\sin x}{2(3\cos x + 2\sin x)}$$

Let $3\cos x + 2\sin x = t$

$(-3\sin x + 2\cos x)dx = dt$

$$\int \frac{2\cos x - 3\sin x}{6\cos x + 4\sin x}dx = \int \frac{dt}{2t}$$

$$= \frac{1}{2}\int \frac{1}{t}dt$$

$$= \frac{1}{2}\log|t| + C$$

$$= \frac{1}{2}\log|2\sin x + 3\cos x| + C$$

Question 25:

$$\frac{1}{\cos^2 x(1-\tan x)^2}$$

Solution:

$$\frac{1}{\cos^2 x(1-\tan x)^2} = \frac{\sec^2 x}{(1-\tan x)^2}$$

Let $(1-\tan x) = t$

$-\sec^2 x\,dx = dt$

$$\Rightarrow \int \frac{\sec^2 x}{(1-\tan x)^2}dx = \int \frac{-dt}{t^2}$$

$$= -\int t^{-2}dt$$

$$= \frac{1}{t} + C$$

$$= \frac{1}{(1-\tan x)} + C$$

Question 26:

$$\frac{\cos \sqrt{x}}{\sqrt{x}}$$

Solution:

Let $\sqrt{x} = t$

$$\frac{1}{2\sqrt{x}} dx = dt$$

$$\Rightarrow \int \frac{\cos \sqrt{x}}{\sqrt{x}} dx = 2 \int \cos t \, dt$$

$$= 2 \sin t + C$$

$$= 2 \sin \sqrt{x} + C$$

Question 27:

$$\sqrt{\sin 2x} \cos 2x$$

Solution:

Put, $\sin 2x = t$

So, $2 \cos 2x \, dx = dt$

$$\Rightarrow \int \sqrt{\sin 2x} \cos 2x \, dx = \frac{1}{2} \int \sqrt{t} \, dt$$

$$= \frac{1}{2} \left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}} \right) + C$$

$$= \frac{1}{3} t^{\frac{3}{2}} + C$$

$$= \frac{1}{3} (\sin 2x)^{\frac{3}{2}} + C$$

Question 28:

$$\frac{\cos x}{\sqrt{1 + \sin x}}$$

Solution:

Put, $1 + \sin x = t$

$$\therefore \cos x \, dx = dt$$

$$\Rightarrow \int \frac{\cos x}{\sqrt{1 + \sin x}} \, dx = \int \frac{dt}{\sqrt{t}}$$

$$= \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= 2\sqrt{t} + C$$

$$= 2\sqrt{1 + \sin x} + C$$

Question 29:

$$\cot x \log \sin x$$

Solution:

Let $\log \sin x = t$

$$\Rightarrow \frac{1}{\sin x} \cos x \, dx = dt$$

$$\therefore \cot x \, dx = dt$$

$$\Rightarrow \int \cot x \log \sin x \, dx = \int t \, dt$$

$$= \frac{t^2}{2} + C$$

$$= \frac{1}{2}(\log \sin x)^2 + C$$

Question 30:

$$\frac{\sin x}{1 + \cos x}$$

Solution:

Put, $1 + \cos x = t$

$$\therefore -\sin x \, dx = dt$$

$$\Rightarrow \int \frac{\sin x}{1 + \cos x} \, dx = \int -\frac{dt}{t}$$

$$= -\log|t| + C$$

$$= -\log|1 + \cos x| + C$$

Question 31:

$$\frac{\sin x}{(1+\cos x)^2}$$

Solution:

Put, $1+\cos x = t$

$\therefore -\sin x \, dx = dt$

$\Rightarrow \int \frac{\sin x}{(1+\cos x)^2} dx = \int -\frac{dt}{t^2}$

$= -\int t^{-2} dt$

$= \frac{1}{t} + C$

$= \frac{1}{(1+\cos x)} + C$

Question 32:

$$\frac{1}{1+\cos x}$$

Solution:

Let $I = \int \frac{1}{1+\cos x} dx$

$= \int \frac{1}{1+\dfrac{\cos x}{\sin x}} dx$

$= \int \frac{\sin x}{\sin x + \cos x} dx$

$= \frac{1}{2} \int \frac{2\sin x}{\sin x + \cos x} dx$

$= \frac{1}{2} \int \frac{(\sin x + \cos x) + (\sin x - \cos x)}{(\sin x + \cos x)} dx$

$= \frac{1}{2} \int 1 dx + \frac{1}{2} \int \frac{\sin x - \cos x}{\sin x + \cos x} dx$

$= \frac{1}{2}(x) + \frac{1}{2} \int \frac{\sin x - \cos x}{\sin x + \cos x} dx$

Let $\sin x + \cos x = t \Rightarrow (\cos x - \sin x) dx = dt$

$\therefore I = \frac{x}{2} + \frac{1}{2} \int \frac{-(dt)}{t}$

$= \frac{x}{2} - \frac{1}{2} \log|t| + C = \frac{x}{2} - \frac{1}{2} \log|\sin x + \cos x| + C$

Question 33:

$$\frac{1}{1-\tan x}$$

Solution:

Put, $I = \int \frac{1}{1-\tan x}dx$

$$= \int \frac{1}{1-\dfrac{\sin x}{\cos x}}dx = \int \frac{\cos x}{\cos x - \sin x}dx$$

$$= \frac{1}{2}\int \frac{2\cos x}{\cos x - \sin x}dx = \frac{1}{2}\int \frac{(\cos x - \sin x)+(\cos x + \sin x)}{(\cos x - \sin x)}dx$$

$$= \frac{1}{2}\int 1 dx + \frac{1}{2}\int \frac{\cos x + \sin x}{\cos x - \sin x}dx = \frac{x}{2} + \frac{1}{2}\int \frac{\cos x + \sin x}{\cos x - \sin x}dx$$

Put, $\cos x - \sin x = t \Rightarrow (-\sin x - \cos x)dx = dt$

$$\therefore I = \frac{x}{2} + \frac{1}{2}\int \frac{-(dt)}{t} = \frac{x}{2} - \frac{1}{2}\log|t| + C$$

$$= \frac{x}{2} - \frac{1}{2}\log|\cos x - \sin x| + C$$

Question 34:

$$\frac{\sqrt{\tan x}}{\sin x \cos x}$$

Solution:

Let $I = \int \frac{\sqrt{\tan x}}{\sin x \cos x}dx = \int \frac{\sqrt{\tan x}\times \cos x}{\sin x \cos x \times \cos x}dx$

$$= \int \frac{\sqrt{\tan x}}{\tan x \cos^2 x}dx = \int \frac{\sec^2 x dx}{\sqrt{\tan x}}$$

Let $\tan x = t \Rightarrow \sec^2 x dx = dt$

$$\therefore I = \int \frac{dt}{\sqrt{t}}$$

$$= 2\sqrt{t} + C$$

$$= 2\sqrt{\tan x} + C$$

Question 35:

$$\frac{(1+\log x)^2}{x}$$

Solution:

Put, $1 + \log x = t$

$\therefore \dfrac{1}{x} dx = dt$

$\Rightarrow \displaystyle\int \dfrac{(1 + \log x)^2}{x} dx = \int t^2 dt$

$= \dfrac{t^3}{3} + C$

$= \dfrac{(1 + \log x)^3}{3} + C$

Question 36:

$\dfrac{(x+1)(x + \log x)^2}{x}$

Solution:

$\dfrac{(x+1)(x + \log x)^2}{x} = \left(\dfrac{x+1}{x}\right)(x + \log)^2 = \left(1 + \dfrac{1}{x}\right)(x + \log x)^2$

Put, $(x + \log x) = t$

$\therefore \left(1 + \dfrac{1}{x}\right) dx = dt$

$\Rightarrow \displaystyle\int \left(1 + \dfrac{1}{x}\right)(x + \log x)^2 dx = \int t^2 dt$

$= \dfrac{t^3}{3} + C$

$= \dfrac{1}{3}(x + \log x)^3 + C$

Question 37:

$\dfrac{x^3 \sin\left(\tan^{-1} x^4\right)}{1 + x^8}$

Solution:

Put, $x^4 = t$

$\therefore 4x^3 dx = dt$

$\Rightarrow \displaystyle\int \dfrac{x^3 \sin\left(\tan^{-1} x^4\right)}{1 + x^8} dx = \dfrac{1}{4}\int \dfrac{\sin\left(\tan^{-1} t\right)}{1 + t^2} dt \qquad \ldots(1)$

Let $\tan^{-1} t = u$

$$\therefore \frac{1}{1+t^2}dt = du$$

From (1), we get

$$\int \frac{x^3 \sin\left(\tan^{-1} x^4\right)dx}{1+x^8} = \frac{1}{4}\int \sin u\, du$$

$$= \frac{1}{4}\left(-\cos u\right)+C$$

$$= -\frac{1}{4}\cos\left(\tan^{-1} t\right)+C$$

$$= \frac{-1}{4}\cos\left(\tan^{-1} x^4\right)+C$$

Choose the correct answer in Exercises 38 and 39.

Question 38:

$$\int \frac{10x^9 + 10^x \log_e 10}{x^{10}+10^x}dx \quad \text{equals}$$

$(A)\; 10^x - x^{10} + C$ $(B)\; 10^x + x^{10} + C$

$(C)\; \left(10^x - x^{10}\right)^{-1} + C$ $(D)\; \log\left(10^x + x^{10}\right)+C$

Solution:

Put, $x^{10} + 10^x = t$

$$\therefore \left(10x^9 + 10^x \log_e 10\right)dx = \int \frac{dt}{t}$$

$$\Rightarrow \int \frac{10x^9 + 10^x \log_e 10}{x^{10}+10x}dx = \int \frac{dt}{t}$$

$$= \log t + C$$

$$= \log\left(10^x + x^{10}\right)+C$$

Thus, the correct option is D.

Question 39:

$$\int \frac{dx}{\sin^2 x \cos^2 x} \quad \text{equals}$$

$(A)\; \tan x + \cot x + C$ $(B)\; \tan x - \cot x + C$

$(C)\; \tan x \cot x + C$ $(D)\; \tan x - \cot 2x + C$

Solution:

Put, $\displaystyle I = \int \frac{dx}{\sin^2 x \cos^2 x} = \int \frac{1}{\sin^2 x \cos^2 x}dx$

$$= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx$$

$$= \int \frac{\sin^2 x}{\sin^2 x \cos^2 x} dx + \frac{\cos^2 x}{\sin^2 x \cos^2 x} dx$$

$$= \int \sec^2 x dx + \int \cos ec^2 dx$$

$$= \tan x - \cot x + C$$

Thus, the correct option is B.

EXERCISE 7.3

Find the integrals of the functions in Exercises 1 to 22:

Question 1:

$\sin^2(2x+5)$

Solution:

$$\sin^2(2x+5) = \frac{1-\cos 2(2x+5)}{2} = \frac{1-\cos(4x+10)}{2}$$

$$\Rightarrow \int \sin^2(2x+5)\,dx = \int \frac{1-\cos(4x+10)}{2}\,dx$$

$$= \frac{1}{2}\int 1\,dx - \frac{1}{2}\int \cos(4x+10)\,dx$$

$$= \frac{1}{2}x - \frac{1}{2}\left(\frac{\sin(4x+10)}{4}\right) + C$$

$$= \frac{1}{2}x - \frac{1}{8}\sin(4x+10) + C$$

Question 2:

$\sin 3x \cos 4x$

Solution:

Using, $\sin A \cos B = \frac{1}{2}\{\sin(A+B) + \sin(A-B)\}$

$$\therefore \int \sin 3x \cos 4x\,dx = \frac{1}{2}\int \{\sin(3x+4x) + \sin(3x-4x)\}\,dx$$

$$= \frac{1}{2}\int \{\sin 7x + \sin(-x)\}\,dx$$

$$= \frac{1}{2}\int \{\sin 7x - \sin x\}\,dx$$

$$= \frac{1}{2}\int \sin 7x\,dx - \frac{1}{2}\int \sin x\,dx$$

$$= \frac{1}{2}\left(\frac{-\cos 7x}{7}\right) - \frac{1}{2}(-\cos x) + C$$

$$= \frac{-\cos 7x}{14} + \frac{\cos x}{2} + C$$

Question 3:

$\cos 2x \cos 4x \cos 6x$

Solution:

Using, $\cos A \cos B = \dfrac{1}{2}\{\cos(A+B)+\cos(A-B)\}$

$\therefore \displaystyle\int \cos 2x(\cos 4x \cos 6x)\,dx = \int \cos 2x\left[\dfrac{1}{2}\{\cos(4x+6x)+\cos(4x-6x)\}\right]dx$

$= \dfrac{1}{2}\displaystyle\int \{\cos 2x \cos 10x + \cos 2x \cos(-2x)\}dx$

$= \dfrac{1}{2}\displaystyle\int \{\cos 2x \cos 10x + \cos^2 2x\}\,dx$

$= \dfrac{1}{2}\displaystyle\int \left[\left\{\dfrac{1}{2}\cos(2x+10x)+\dfrac{1}{2}\cos(2x-10x)\right\}+\left(\dfrac{1+\cos 4x}{2}\right)\right]dx$

$= \dfrac{1}{4}\displaystyle\int (\cos 12x + \cos 8x + 1 + \cos 4x)\,dx$

$= \dfrac{1}{4}\left[\dfrac{\sin 12x}{12}+\dfrac{\sin 8x}{8}+x+\dfrac{\sin 4x}{4}+C\right]$

Question 4:

$\sin^3(2x+1)$

Solution:

Put, $I = \displaystyle\int \sin^3(2x+1)$

$\Rightarrow \displaystyle\int \sin^3(2x+1)\,dx = \int \sin^2(2x+1)\sin(2x+1)\,dx$

$= \displaystyle\int \left(1-\cos^2(2x+1)\right)\sin(2x+1)\,dx$

Let $\cos(2x+1) = t$

$\Rightarrow -2\sin(2x+1)\,dx = dt$

$\Rightarrow \sin(2x+1)\,dx = \dfrac{-dt}{2}$

$\Rightarrow I = \dfrac{-1}{2}\displaystyle\int \left(1-t^2\right)dt$

$= \dfrac{-1}{2}\left\{t-\dfrac{t^3}{3}\right\}$

$= \dfrac{-1}{2}\left\{\cos(2x+1)-\dfrac{\cos^3(2x+1)}{3}\right\}$

$= \dfrac{-\cos(2x+1)}{2}+\dfrac{\cos^3(2x+1)}{6}+C$

$\sin^3 x \cos^3 x$

Solution:

Let $I = \int \sin^3 x \cos^3 x \, dx$

$\quad = \int \cos^3 x \sin^2 x \sin x \, dx$

$\quad = \int \cos^3 x \left(1 - \cos^2 x\right) \sin x \, dx$

Let $\cos x = t$

$\Rightarrow -\sin x \, dx = dt$

$\Rightarrow I = -\int t^3 \left(1 - t^2\right) dt$

$\quad = -\int \left(t^3 - t^5\right) dt = -\left\{\dfrac{t^4}{4} - \dfrac{t^6}{6}\right\} + C$

$\quad = -\left\{\dfrac{\cos^4 x}{4} - \dfrac{\cos^6 x}{6}\right\} + C = \dfrac{\cos^6 x}{6} - \dfrac{\cos^4 x}{4} + C$

Question 6:

$\sin x \sin 2x \sin 3x$

Solution:

Using, $\sin A \sin B = \dfrac{1}{2}\left\{\cos\left(A - B\right) - \cos\left(A + B\right)\right\}$

$\therefore \int \sin x \sin 2x \sin 3x \, dx = \int \left[\sin x \dfrac{1}{2}\left\{\cos\left(2x - 3x\right) - \cos\left(2x + 3x\right)\right\}\right] dx$

$\quad = \dfrac{1}{2} \int \left(\sin x \cos\left(-x\right) - \sin x \cos 5x\right) dx$

$\quad = \dfrac{1}{2} \int \left(\sin x \cos x - \sin x \cos 5x\right) dx$

$\quad = \dfrac{1}{2} \int \dfrac{\sin 2x}{2} \, dx - \dfrac{1}{2} \int \sin x \cos 5x \, dx$

$\quad = \dfrac{1}{4}\left[\dfrac{-\cos 2x}{2}\right] - \dfrac{1}{2} \int \left\{\dfrac{1}{2}\sin\left(x + 5x\right) + \dfrac{1}{2}\sin\left(x - 5x\right)\right\} dx$

$\quad = \dfrac{-\cos 2x}{8} - \dfrac{1}{4} \int \left(\sin 6x + \sin\left(-4x\right)\right) dx$

$\quad = \dfrac{-\cos 2x}{8} - \dfrac{1}{4}\left[\dfrac{-\cos 6x}{6} + \dfrac{\cos 4x}{4}\right] + C$

$\quad = \dfrac{-\cos 2x}{8} - \dfrac{1}{8}\left[\dfrac{-\cos 6x}{3} + \dfrac{\cos 4x}{2}\right] + C$

$\quad = \dfrac{1}{8}\left[\dfrac{\cos 6x}{3} - \dfrac{\cos 4x}{2} - \cos 2x\right] + C$

Question 7:

$\sin 4x \sin 8x$

Solution:

Using, $\sin A \sin B = \dfrac{1}{2}\{\cos(A-B) - \cos(A+B)\}$

$$\therefore \int \sin 4x \sin 8x \, dx = \int \left\{ \dfrac{1}{2}\cos(4x - 8x) - \dfrac{1}{2}\cos(4x + 8x) \right\} dx$$

$$= \dfrac{1}{2}\int (\cos(-4x) - \cos 12x) \, dx$$

$$= \dfrac{1}{2}\int (\cos 4x - \cos 12x) \, dx$$

$$= \dfrac{1}{2}\left[\dfrac{\sin 4x}{4} - \dfrac{\sin 12x}{12} \right]$$

Question 8:

$\dfrac{1 - \cos x}{1 + \cos x}$

Solution:

$$\dfrac{1 - \cos x}{1 + \cos x} = \dfrac{2\sin^2 \dfrac{x}{2}}{2\cos^2 \dfrac{x}{2}} \qquad \left[2\sin^2 \dfrac{x}{2} = 1 - \cos x \text{ and } 2\cos^2 \dfrac{x}{2} = 1 + \cos x \right]$$

$$= \tan^2 \dfrac{x}{2}$$

$$= \left(\sec^2 \dfrac{x}{2} - 1 \right)$$

$$\therefore \dfrac{1 - \cos x}{1 + \cos x} \, dx = \int \left(\sec^2 \dfrac{x}{2} - 1 \right) dx$$

$$= \left[\dfrac{\tan \dfrac{x}{2}}{\dfrac{1}{2}} - x \right] + C$$

$$= 2\tan \dfrac{x}{2} - x + C$$

Question 9:

$$\frac{\cos x}{1+\cos x}$$

Solution:

$$\frac{\cos x}{1+\cos x} = \frac{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}{2\cos^2 \frac{x}{2}} \qquad \left[\cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} \text{ and } \cos x = 2\cos^2 \frac{x}{2} - 1 \right]$$

$$= \frac{1}{2}\left[1 - \tan^2 \frac{x}{2} \right]$$

$$\therefore \int \frac{\cos x}{1+\cos x}\,dx = \frac{1}{2}\int \left(1 - \tan^2 \frac{x}{2} \right)dx$$

$$= \frac{1}{2}\int \left(1 - \sec^2 \frac{x}{2} + 1 \right)dx$$

$$= \frac{1}{2}\int \left(2 - \sec^2 \frac{x}{2} \right)dx$$

$$= \frac{1}{2}\left[2x - \frac{\tan \frac{x}{2}}{\frac{1}{2}} \right] + C$$

$$= x - \tan \frac{x}{2} + C$$

Question 10:

$$\sin^4 x$$

Solution:

$$\sin^4 x = \sin^2 x \sin^2 x$$

$$= \left(\frac{1 - \cos 2x}{2} \right) \left(\frac{1 - \cos 2x}{2} \right)$$

$$= \frac{1}{4} (1 - \cos 2x)^2$$

$$= \frac{1}{4} \left[1 + \cos^2 2x - 2 \cos 2x \right]$$

$$= \frac{1}{4} \left[1 + \left(\frac{1 + \cos 4x}{2} \right) - 2 \cos 2x \right]$$

$$= \frac{1}{4} \left[1 + \frac{1}{2} + \frac{1}{2} \cos 4x - 2 \cos 2x \right]$$

$$= \frac{1}{4} \left[\frac{3}{2} + \frac{1}{2} \cos 4x - 2 \cos 2x \right]$$

$$\therefore \int \sin^4 x \, dx = \frac{1}{4} \int \left[\frac{3}{2} + \frac{1}{2} \cos 4x - 2 \cos 2x \right] dx$$

$$= \frac{1}{4} \left[\frac{3}{2} x + \frac{1}{2} \left(\frac{\sin 4x}{4} \right) - 2 \times \frac{\sin 2x}{2} \right] + C$$

$$= \frac{1}{8} \left[3x + \frac{\sin 4x}{4} - 2 \sin 2x \right] + C$$

$$= \frac{3x}{8} - \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + C$$

Question 11:

$$\cos^4 2x$$

Solution:

$$\cos^4 2x = \left(\cos^2 2x\right)^2$$

$$= \left(\frac{1 + \cos 4x}{2}\right)^2$$

$$= \frac{1}{4}\left[1 + \cos^2 4x + 2\cos 4x\right]$$

$$= \frac{1}{4}\left[1 + \left(\frac{1 + \cos 8x}{2}\right) + 2\cos 4x\right]$$

$$= \frac{1}{4}\left[1 + \frac{1}{2} + \frac{\cos 8x}{2} + 2\cos 4x\right]$$

$$= \frac{1}{4}\left[\frac{3}{2} + \frac{\cos 8x}{2} + 2\cos 4x\right]$$

$$\therefore \int \cos^4 2x \, dx = \int \left(\frac{3}{8} + \frac{\cos 8x}{8} + \frac{\cos 4x}{2}\right) dx$$

$$= \frac{3}{8}x + \frac{1}{64}\sin 8x + \frac{1}{8}\sin 4x + C$$

Question 12:

$$\frac{\sin^2 x}{1 + \cos x}$$

Solution:

$$\frac{\sin^2 x}{1 + \cos x} = \frac{\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)^2}{2\cos^2\frac{x}{2}} \qquad \left[\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}; \cos x = 2\cos^2\frac{x}{2} - 1\right]$$

$$= \frac{4\sin^2\frac{x}{2}\cos^2\frac{x}{2}}{2\cos^2\frac{x}{2}}$$

$$= 2\sin^2\frac{x}{2}$$

$$= 1 - \cos x$$

$$\therefore \int \frac{\sin^2 x}{1 + \cos x} dx = \int (1 - \cos x) dx$$

$$= x - \sin x + C$$

Question 13:

$$\frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}$$

Solution:

$$\frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} = \frac{-2\sin\dfrac{2x + 2\alpha}{2}\sin\dfrac{2x - 2\alpha}{2}}{-2\sin\dfrac{x+\alpha}{2}\sin\dfrac{x-\alpha}{2}} \qquad \left[\cos C - \cos D = -2\sin\dfrac{C+D}{2}\sin\dfrac{C-D}{2}\right]$$

$$= \frac{\sin(x+\alpha)\sin(x-\alpha)}{\sin\left(\dfrac{x+\alpha}{2}\right)\sin\left(\dfrac{x-\alpha}{2}\right)}$$

$$= \frac{\left[2\sin\left(\dfrac{x+\alpha}{2}\right)\cos\left(\dfrac{x-\alpha}{2}\right)\right]\left[2\sin\left(\dfrac{x-\alpha}{2}\right)\cos\left(\dfrac{x+\alpha}{2}\right)\right]}{\sin\left(\dfrac{x+\alpha}{2}\right)\sin\left(\dfrac{x-\alpha}{2}\right)}$$

$$= 4\cos\left(\dfrac{x+\alpha}{2}\right)\cos\left(\dfrac{x-\alpha}{2}\right)$$

$$= 2\left[\cos\left(\dfrac{x+\alpha}{2} + \dfrac{x-\alpha}{2}\right) + \cos\left(\dfrac{x+\alpha}{2} - \dfrac{x-\alpha}{2}\right)\right]$$

$$= 2\left[\cos(x) + \cos\alpha\right]$$

$$= 2\cos x + 2\cos\alpha$$

$$\therefore \int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}\,dx = \int 2\cos x + 2\cos\alpha\,dx$$

$$= 2\left[\sin x + x\cos\alpha\right] + C$$

Question 14:

$$\frac{\cos x - \sin x}{1 + \sin 2x}$$

Solution:

$$\frac{\cos x - \sin x}{1 + \sin 2x} = \frac{\cos x - \sin x}{(\sin^2 x + \cos^2 x) + 2\sin x\cos x} \qquad \left[\sin^2 x + \cos^2 x = 1; \sin 2x = 2\sin x\cos x\right]$$

$$= \frac{\cos x - \sin x}{(\sin x + \cos x)^2}$$

Let $\sin x + \cos x = t$

$$\therefore \left(\cos x - \sin x\right) dx = dt$$

$$\Rightarrow \int \frac{\cos x - \sin x}{1 + \sin 2x} dx = \int \frac{\cos x - \sin x}{\left(\sin x + \cos x\right)^2} dx$$

$$= \int \frac{dt}{t^2}$$

$$= \int t^{-2} dt$$

$$= -t^{-1} + C$$

$$= -\frac{1}{t} + C$$

$$= \frac{-1}{\sin x + \cos x} + C$$

Question 15:

$\tan^3 2x \sec 2x$

Solution:

$$\tan^3 2x \sec 2x = \tan^2 2x \tan 2x \sec 2x$$

$$= \left(\sec^2 2x - 1\right) \tan 2x \sec 2x$$

$$= \sec^2 2x \tan 2x \sec 2x - \tan 2x \sec 2x$$

$$\therefore \int \tan^3 2x \sec 2x\, dx = \int \sec^2 2x \tan 2x \sec 2x - \int \tan 2x \sec 2x$$

$$= \int \sec^2 2x \tan 2x \sec 2x - \frac{\sec 2x}{2} + C$$

Let $\sec 2x = t$

$$\therefore 2 \sec 2x \tan 2x\, dx = dt$$

$$\therefore \int \tan^3 2x \sec 2x\, dx = \frac{1}{2} \int t^2 dt - \frac{\sec 2x}{2} + C$$

$$= \frac{t^3}{6} - \frac{\sec 2x}{2} + C$$

$$= \frac{\left(\sec 2x\right)^3}{6} - \frac{\sec 2x}{2} + C$$

Question 16:

$\tan^4 x$

Solution:

$\tan^4 x$

$= \tan^2 x \tan^2 x$

$= \left(\sec^2 x - 1\right)\tan^2 x$

$= \sec^2 x \tan^2 x - \tan^2 x$

$= \sec^2 x \tan^2 x - \left(\sec^2 x - 1\right)$

$= \sec^2 x \tan^2 x - \sec^2 x + 1$

$\therefore \int \tan^4 x\, dx = \int \sec^2 x \tan^2 x\, dx - \int \sec^2 x\, dx + \int 1\, dx$

$\qquad\qquad = \int \sec^2 x \tan^2 x\, dx - \tan x + x + C \quad \ldots(1)$

Consider $\sec^2 x \tan^2 x\, dx$

Let $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$\Rightarrow \int \sec^2 x \tan^2 x\, dx = \int t^2\, dt = \dfrac{t^3}{3} = \dfrac{\tan^3 x}{3}$

From equation (1), we get

$\int \tan^4 x\, dx = \dfrac{1}{3}\tan^3 x - \tan x + x + C$

Question 17:

$\dfrac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}$

Solution:

$\dfrac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} = \dfrac{\sin^3 x}{\sin^2 x \cos^2 x} + \dfrac{\cos^3 x}{\sin^2 x \cos^2 x}$

$\qquad\qquad = \dfrac{\sin x}{\cos^2 x} + \dfrac{\cos x}{\sin^2 x}$

$\qquad\qquad = \tan x \sec x + \cot x \,\operatorname{cosec} x$

$\therefore \int \dfrac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}\, dx = \int \left(\tan x \sec x + \cot x \,\operatorname{cosec} x\right) dx$

$\qquad\qquad = \sec x - \operatorname{cosec} x + C$

$$\frac{\cos 2x + 2\sin^2 x}{\cos^2 x}$$

Solution:

$$\frac{\cos 2x + 2\sin^2 x}{\cos^2 x}$$

$$= \frac{\cos 2x + (1 - \cos 2x)}{\cos^2 x} \quad \left[\cos 2x = 1 - 2\sin^2 x\right]$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

$$\therefore \int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x} dx = \int \sec^2 x\, dx = \tan x + C$$

Question 19:

$$\frac{1}{\sin x \cos^3 x}$$

Solution:

$$\frac{1}{\sin x \cos^3 x} = \frac{\sin^2 x + \cos^2 x}{\sin x \cos^3 x}$$

$$= \frac{\sin x}{\cos^3 x} + \frac{1}{\sin x \cos x}$$

$$= \tan x \sec^2 x + \frac{\dfrac{1}{\cos^2 x}}{\dfrac{\sin x \cos x}{\cos^2 x}}$$

$$= \tan x \sec^2 x + \frac{\sec^2 x}{\tan x}$$

$$\therefore \int \frac{1}{\sin x \cos^3 x} dx = \int \tan x \sec^2 x\, dx + \int \frac{\sec^2 x}{\tan x} dx$$

Let $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$$\Rightarrow \int \frac{1}{\sin x \cos^3 x} dx = \int t\, dt + \int \frac{1}{t} dt$$

$$= \frac{t^2}{2} + \log|t| + C$$

$$= \frac{1}{2}\tan^2 x + \log|\tan x| + C$$

Question 20:

$$\frac{\cos 2x}{(\cos x + \sin x)^2}$$

Solution:

$$\frac{\cos 2x}{(\cos x + \sin x)^2} = \frac{\cos 2x}{\cos^2 x + \sin x^2 + 2\cos x \sin x} = \frac{\cos 2x}{1 + \sin 2x}$$

$$\therefore \int \frac{\cos 2x}{(\cos x + \sin x)^2} \, dx = \int \frac{\cos 2x}{(1 + \sin 2x)} \, dx$$

Let $1 + \sin 2x = t$

$$\Rightarrow 2\cos 2x \, dx = dt$$

$$\therefore \int \frac{\cos 2x}{(\cos x + \sin x)^2} \, dx = \frac{1}{2} \int \frac{1}{t} \, dt$$

$$= \frac{1}{2} \log |t| + C$$

$$= \frac{1}{2} \log |1 + \sin 2x| + C$$

$$= \frac{1}{2} \log \left| (\sin x + \cos x)^2 \right| + C$$

$$= \log |\sin x + \cos x| + C$$

Question 21:

$$\sin^{-1}(\cos x)$$

Solution:

$$\sin^{-1}(\cos x)$$

Let $\cos x = t$

Then, $\sin x = \sqrt{1 - t^2}$

$$\Rightarrow (-\sin x) \, dx = dt$$

$$dx = \frac{-dt}{\sin x}$$

$$dx = \frac{-dt}{\sqrt{1 - t^2}}$$

$$\therefore \int \sin^{-1}(\cos x) \, dx = \int \sin^{-1} t \left(\frac{-dt}{\sqrt{1 - t^2}} \right)$$

$$= -\int \frac{\sin^{-1} t}{\sqrt{1 - t^2}}$$

Let $\sin^{-1} t = u$

$\Rightarrow \dfrac{1}{\sqrt{1-t^2}}\,dt = du$

$\therefore \displaystyle\int \sin^{-1}(\cos x)\,dx = -\int u\,du$

$= -\dfrac{u^2}{2} + C$

$= \dfrac{-\left(\sin^{-1} t\right)^2}{2} + C$

$= \dfrac{-\left[\sin^{-1}(\cos x)\right]^2}{2} + C \quad \ldots(1)$

We know that,

$\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}$

$\therefore \sin^{-1}(\cos x) = \dfrac{\pi}{2} - \cos^{-1}(\cos x) = \left(\dfrac{\pi}{2} - x\right)$

Substituting in equation (1), we get

$\displaystyle\int \sin^{-1}(\cos x)\,dx = \dfrac{-\left[\dfrac{\pi}{2} - x\right]^2}{2} + C$

$= -\dfrac{1}{2}\left(\dfrac{\pi^2}{4} + x^2 - \pi x\right) + C$

$= -\dfrac{\pi^2}{8} - \dfrac{x^2}{2} + \dfrac{\pi x}{2} + C$

$= \dfrac{\pi x}{2} - \dfrac{x^2}{2} + \left(C - \dfrac{\pi^2}{8}\right)$

$= \dfrac{\pi x}{2} - \dfrac{x^2}{2} + C_1$

Question 22:

$$\frac{1}{\cos(x-a)\cos(x-b)}$$

Solution:

$$\frac{1}{\cos(x-a)\cos(x-b)} = \frac{1}{\sin(a-b)}\left[\frac{\sin(a-b)}{\cos(x-a)\cos(x-b)}\right]$$

$$= \frac{1}{\sin(a-b)}\left[\frac{\sin[(x-b)-(x-a)]}{\cos(x-a)\cos(x-b)}\right]$$

$$= \frac{1}{\sin(a-b)}\left[\frac{\sin(x-b)\cos(x-a)-\cos(x-b)\sin(x-a)}{\cos(x-a)\cos(x-b)}\right]$$

$$= \frac{1}{\sin(a-b)}\left[\tan(x-b)-\tan(x-a)\right]$$

$$\Rightarrow \int\frac{1}{\cos(x-a)\cos(x-b)}dx = \frac{1}{\sin(a-b)}\int\left[\tan(x-b)-\tan(x-a)\right]dx$$

$$= \frac{1}{\sin(a-b)}\left[-\log|\cos(x-b)|+\log|\cos(x-a)|\right]$$

$$= \frac{1}{\sin(a-b)}\left[\log\left|\frac{\cos(x-a)}{\cos(x-b)}\right|\right]+C$$

Choose the correct answer in Exercises 23 and 24.

Question 23:

$$\int\frac{\sin^2 x-\cos^2 x}{\sin^2 x\cos^2 x}dx \text{ is equal to}$$

$(A)\ \tan x+\cot x+C$ $\qquad$ $(B)\ \tan x+\cos ecx+C$

$(C)\ -\tan x+\cot x+C$ $\qquad$ $(D)\ \tan x+\sec x+C$

Solution:

$$\int\frac{\sin^2 x-\cos^2 x}{\sin^2 x\cos^2 x}dx = \int\left(\frac{\sin^2 x}{\sin^2 x\cos^2 x}-\frac{\cos^2 x}{\sin^2 x\cos^2 x}\right)dx$$

$$= \int\left(\sec^2 x-\cos ec^2 x\right)dx$$

$$= \tan x+\cot x+C$$

Thus, the correct option is A.

Question 24:

$\int \dfrac{e^x (1+x)}{\cos^2 (e^x x)} dx$ equals

(A) $-\cot (ex^x) + C$ 　　　(B) $\tan (xe^x) + C$

(C) $\tan (e^x) + C$ 　　　(D) $\cot (e^x) + C$

Solution:

$\int \dfrac{e^x (1+x)}{\cos^2 (e^x x)} dx$

Put, $e^x x = t$

$\Rightarrow (e^x x + e^x .1) dx = dt$

$e^x (x+1) dx = dt$

$\therefore \int \dfrac{e^x (1+x)}{\cos^2 (e^x x)} dx = \int \dfrac{dt}{\cos^2 t}$

$= \int \sec^2 t\, dt$

$= \tan t + C$

$= \tan (e^x x) + C$

Thus, the correct answer is B.

EXERCISE 7.4

Integrate the functions in Exercises 1 to 23

Question 1:

$$\frac{3x^2}{x^6 + 1}$$

Solution:

Put, $x^3 = t$

$\therefore 3x^2 dx = dt$

$$\Rightarrow \frac{3x^2}{x^6 + 1} dx = \int \frac{dt}{t^2 + 1}$$

$$= \tan^{-1} t + C$$

$$= \tan^{-1}(x^3) + C$$

Question 2:

$$\frac{1}{\sqrt{1 + 4x^2}}$$

Solution:

Put, $2x = t$

$\therefore 2dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{1 + 4x^2}} dx = \frac{1}{2} \int \frac{dt}{\sqrt{1 + t^2}}$$

$$= \frac{1}{2} \left[\log \left| t + \sqrt{t^2 + 1} \right| \right] + C \qquad \left[\int \frac{1}{\sqrt{x^2 + a^2}} dt = \log \left| x + \sqrt{x^2 + a^2} \right| \right]$$

$$= \frac{1}{2} \log \left| 2x + \sqrt{4x^2 + 1} \right| + C$$

Question 3:

$$\frac{1}{\sqrt{(2 - x)^2 + 1}}$$

Solution:

Put, $2 - x = t$

$$\Rightarrow -dx = dt$$

$$\Rightarrow \int \frac{1}{\sqrt{(2-x)^2+1}}dx = -\int \frac{1}{\sqrt{t^2+1}}dt$$

$$= -\log\left|t+\sqrt{t^2+1}\right|+C \qquad \left[\int \frac{1}{\sqrt{x^2+a^2}}dt = \log\left|x+\sqrt{x^2+a^2}\right|\right]$$

$$= -\log\left|2-x+\sqrt{(2-x)^2+1}\right|+C$$

$$= \log\left|\frac{1}{(2-x)+\sqrt{x^2-4x+5}}\right|+C$$

Question 4:

$$\frac{1}{\sqrt{9-25x^2}}$$

Solution:

Put, $5x = t$

$$\therefore 5dx = dt$$

$$\Rightarrow \int \frac{1}{\sqrt{9-25x^2}}dx = \frac{1}{5}\int \frac{1}{\sqrt{9-t^2}}dt$$

$$= \frac{1}{5}\int \frac{1}{\sqrt{3^2-t^2}}dt$$

$$= \frac{1}{5}\sin^{-1}\left(\frac{t}{3}\right)+C$$

$$= \frac{1}{5}\sin^{-1}\left(\frac{5x}{3}\right)+C$$

Question 5:

$$\frac{3x}{1+2x^4}$$

Solution:

Let $\sqrt{2}x^2 = t$

$$\therefore 2\sqrt{2}xdx = dt$$

$$\Rightarrow \int \frac{3x}{1+2x^4}dx = \frac{3}{2\sqrt{2}}\int \frac{dt}{1+t^2}$$

$$= \frac{3}{2\sqrt{2}}\left[\tan^{-1}t\right] + C$$

$$= \frac{3}{2\sqrt{2}}\tan^{-1}\left(\sqrt{2}x^2\right) + C$$

Question 6:

$$\frac{x^2}{1-x^6}$$

Solution:

Put, $x^3 = t$

$$\therefore 3x^2 dx = dt$$

$$\Rightarrow \int \frac{x^2}{1-x^6}dx = \frac{1}{3}\int \frac{dt}{1-t^2}$$

$$= \frac{1}{3}\left[\frac{1}{2}\log\left|\frac{1+t}{1-t}\right|\right] + C$$

$$= \frac{1}{6}\log\left|\frac{1+x^3}{1-x^3}\right| + C$$

Question 7:

$$\frac{x-1}{\sqrt{x^2-1}}$$

Solution:

$$\int \frac{x-1}{\sqrt{x^2-1}}dx = \int \frac{x}{\sqrt{x^2-1}}dx - \int \frac{1}{\sqrt{x^2-1}}dx \quad \ldots(1)$$

For $\int \frac{x}{\sqrt{x^2-1}}dx,$ let $x^2-1 = t \Rightarrow 2xdx = dt$

$$\therefore \int \frac{x}{\sqrt{x^2-1}} dx = \frac{1}{2} \int \frac{dt}{\sqrt{t}}$$

$$= \frac{1}{2} \int t^{-\frac{1}{2}} dt$$

$$= \frac{1}{2} \left[2t^{\frac{1}{2}} \right]$$

$$= \sqrt{t}$$

$$= \sqrt{x^2-1}$$

From (1), we get

$$\int \frac{x-1}{\sqrt{x^2-1}} dx = \int \frac{x}{\sqrt{x^2-1}} dx - \int \frac{1}{\sqrt{x^2-1}} dx \qquad \left[\int \frac{x}{\sqrt{x^2-a^2}} dt = \log \left| x + \sqrt{x^2-a^2} \right| \right]$$

$$= \sqrt{x^2-1} - \log \left| x + \sqrt{x^2-1} \right| + C$$

Question 8:

$$\frac{x^2}{\sqrt{x^6+a^6}}$$

Solution:

Put, $x^3 = t \Rightarrow 3x^2 dx = dt$

$$\therefore \int \frac{x^2}{\sqrt{x^6+a^6}} dx = \frac{1}{3} \int \frac{dt}{\sqrt{t^2 + \left(a^3\right)^2}}$$

$$= \frac{1}{3} \log \left| t + \sqrt{t^2+a^6} \right| + C$$

$$= \frac{1}{3} \log \left| x^3 + \sqrt{x^6+a^6} \right| + C$$

Question 9:

$$\frac{\sec^2 x}{\sqrt{\tan^2 x + 4}}$$

Solution:

Put, $\tan x = t$

$$\therefore \sec^2 x \, dx = dt$$

$$\Rightarrow \int \frac{\sec^2 x}{\sqrt{\tan^2 x + 4}} \, dx = \int \frac{dt}{\sqrt{t^2 + 2^2}}$$

$$= \log\left| t + \sqrt{t^2 + 4} \right| + C$$

$$= \log\left| \tan x + \sqrt{\tan^2 x + 4} \right| + C$$

Question 10:

$$\frac{1}{\sqrt{x^2 + 2x + 2}}$$

Solution:

$$\int \frac{1}{\sqrt{x^2 + 2x + 2}} \, dx = \int \frac{1}{\sqrt{(x+1)^2 + (1)^2}} \, dx$$

Let $x + 1 = t$

$$\therefore dx = dt$$

$$\Rightarrow \int \frac{1}{\sqrt{x^2 + 2x + 2}} \, dx = \int \frac{1}{\sqrt{t^2 + 1}} \, dt$$

$$= \log\left| t + \sqrt{t^2 + 1} \right| C$$

$$= \log\left| (x+1) + \sqrt{(x+1)^2 + 1} \right| + C$$

$$= \log\left| (x+1) + \sqrt{x^2 + 2x + 2} \right| + C$$

Question 11:

$$\frac{1}{\sqrt{9x^2 + 6x + 5}}$$

Solution:

$$\int \frac{1}{\sqrt{9x^2+6x+5}} dx = \int \frac{1}{(3x+1)^2+(2)^2} dx$$

Let $(3x+1) = t$

$\Rightarrow 3dx = dt$

$$\Rightarrow \int \frac{1}{(3x+1)^2+(2)^2} dx = \frac{1}{3}\int \frac{1}{t^2+2^2} dt$$

$$= \frac{1}{3}\left[\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)\right]+C$$

$$= \frac{1}{6}\left[\tan^{-1}\left(\frac{3x+1}{2}\right)\right]+C$$

Question 12:

$$\frac{1}{\sqrt{7-6x-x^2}}$$

Solution:

$7-6x-x^2$ can be written as $7-\left(x^2+6x+9-9\right)$

Thus,

$7-\left(x^2+6x+9-9\right)$

$=16-\left(x^2+6x+9\right)$

$=16-\left(x+3\right)^2$

$=(4)^2-(x+3)^2$

$$\therefore \int \frac{1}{\sqrt{7-6x-x^2}} dx = \int \frac{1}{\sqrt{(4)^2-(x+3)^2}} dx$$

Let $x+3 = t$

$\Rightarrow dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{(4)^2-(x+3)^2}} dx = \int \frac{1}{\sqrt{(4)^2-(t)^2}} dt$$

$$= \sin^{-1}\left(\frac{t}{4}\right)+C$$

$$= \sin^{-1}\left(\frac{x+3}{4}\right)+C$$

Question 13:

$$\frac{1}{\sqrt{(x-1)(x-2)}}$$

Solution:

$(x-1)(x-2)$ can be written as $x^2 - 3x + 2$

Thus,

$$x^2 - 3x + 2$$
$$= x^2 - 3x + \frac{9}{4} - \frac{9}{4} + 2$$
$$= \left(x - \frac{3}{2}\right)^2 - \frac{1}{4}$$
$$= \left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2$$

$$\therefore \int \frac{1}{\sqrt{(x-1)(x-2)}} dx = \int \frac{1}{\sqrt{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} dx$$

Let $\left(x - \frac{3}{2}\right) = t$

$$\therefore dx = dt$$

$$\Rightarrow \int \frac{1}{\sqrt{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} dx = \int \frac{1}{\sqrt{t^2 - \left(\frac{1}{2}\right)^2}} dt$$

$$= \log \left| t + \sqrt{t^2 - \left(\frac{1}{2}\right)^2} \right| + C$$

$$= \log \left| \left(x - \frac{3}{2}\right) + \sqrt{x^2 - 3x + 2} \right| + C$$

Question 14:

$$\frac{1}{\sqrt{8 + 3x - x^2}}$$

Solution:

$$8 + 3x - x^2 = 8 - \left(x^2 - 3x + \frac{9}{4} - \frac{9}{4}\right)$$

Thus,

$$8 - \left(x^2 - 3x + \frac{9}{4} - \frac{9}{4} \right)$$

$$= \frac{41}{4} - \left(x - \frac{3}{2} \right)^2$$

$$= \int \frac{1}{\sqrt{8 + 3x - x^2}} \, dx = \int \frac{1}{\sqrt{\frac{41}{4} - \left(x - \frac{3}{2} \right)^2}} \, dx$$

Let $\left(x - \frac{3}{2} \right) = t$

$\therefore dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{\frac{41}{4} - \left(x - \frac{3}{2} \right)^2}} \, dx = \int \frac{1}{\sqrt{\left(\frac{41}{4} \right) - t^2}} \, dt$$

$$= \sin^{-1} \left(\frac{t}{\frac{\sqrt{41}}{2}} \right) + C$$

$$= \sin^{-1} \left(\frac{x - \frac{3}{2}}{\frac{\sqrt{41}}{2}} \right) + C$$

$$= \sin^{-1} \left(\frac{2x - 3}{\sqrt{41}} \right) + C$$

Question 15:

$$\frac{1}{\sqrt{(x - a)(x - b)}}$$

Solution:

$(x - a)(x - b) = x^2 - (a + b)x + ab$

Thus,

$x^2 - (a+b)x + ab$

$= x^2 - (a+b)x + \dfrac{(a+b)^2}{4} - \dfrac{(a+b)^2}{4} + ab$

$= \left[x - \left(\dfrac{a+b}{2} \right) \right]^2 - \dfrac{(a-b)^2}{4}$

$\Rightarrow \displaystyle\int \dfrac{1}{\sqrt{(x-a)(x-b)}}\, dx = \int \dfrac{1}{\sqrt{\left\{ x - \left(\dfrac{a+b}{2} \right) \right\}^2 - \left(\dfrac{a+b}{2} \right)^2}}\, dx$

Let $x - \left(\dfrac{a+b}{2} \right) = t$

$\therefore dx = dt$

$\Rightarrow \displaystyle\int \dfrac{1}{\sqrt{\left\{ x - \left(\dfrac{a+b}{2} \right) \right\}^2 - \left(\dfrac{a+b}{2} \right)^2}}\, dx = \int \dfrac{1}{\sqrt{t^2 - \left(\dfrac{a+b}{2} \right)^2}}\, dt$

$= \log \left| t + \sqrt{t^2 - \left(\dfrac{a+b}{2} \right)^2} \right| + C$

$= \log \left| \left\{ x - \left(\dfrac{a+b}{2} \right) \right\} + \sqrt{(x-a)(x-b)} \right| + C$

Question 16:

$\dfrac{4x+1}{\sqrt{2x^2 + x - 3}}$

Solution:

Let, $4x + 1 = A \dfrac{d}{dx}\left(2x^2 + x - 3 \right) + B$

$\Rightarrow 4x + 1 = A(4x + 1) + B$

$\Rightarrow 4x + 1 = 4Ax + A + B$

Equating the coefficients of x and constant term on both sides, we get

$4A = 4 \Rightarrow A = 1$

$A + B = 1 \Rightarrow B = 0$

Let $2x^2 + x - 3 = t$

$\therefore (4x+1)\, dx = dt$

$\Rightarrow \displaystyle\int \dfrac{4x+1}{\sqrt{2x^2 + x - 3}}\, dx = \int \dfrac{1}{\sqrt{t}}\, dt$

$= 2\sqrt{t} + C$

$= 2\sqrt{2x^2 + x - 3} + C$

Question 17:

$$\frac{x+2}{\sqrt{x^2-1}}$$

Solution:

Put, $x+2 = A\dfrac{d}{dx}\left(x^2-1\right)+B$ ...(1)

$\Rightarrow x+2 = A(2x)+B$

Equating the coefficients of x and constant term on both sides, we get

$2A = 1 \Rightarrow A = \dfrac{1}{2}$

$B = 2$

From (1) we get

$\Rightarrow \displaystyle\int \frac{x+2}{\sqrt{x^2-1}}dx = \int \frac{\dfrac{1}{2}(2x)+2}{\sqrt{x^2-1}}dx$

$= \dfrac{1}{2}\displaystyle\int \frac{2x}{\sqrt{x^2-1}}dx + \int \frac{2}{\sqrt{x^2-1}}dx$ ...(2)

In $\dfrac{1}{2}\displaystyle\int \frac{2x}{\sqrt{x^2-1}}dx$, Let $x^2-1 = t \Rightarrow 2xdx = dt$

$\dfrac{1}{2}\displaystyle\int \frac{2x}{\sqrt{x^2-1}}dx = \dfrac{1}{2}\int \frac{dt}{\sqrt{t}}$

$= \dfrac{1}{2}\left[2\sqrt{t}\right]$

$= \dfrac{1}{2}\left[2\sqrt{x^2-1}\right]$

$= \sqrt{x^2-1}$

Then, $\displaystyle\int \frac{2}{\sqrt{x^2-1}}dx = 2\int \frac{1}{\sqrt{x^2-1}}dx = 2\log\left|x+\sqrt{x^2-1}\right|$

From equation (2) we get

$\displaystyle\int \frac{x+2}{\sqrt{x^2-1}}dx = \sqrt{x^2-1} + 2\log\left|x+\sqrt{x^2-1}\right| + C$

Question 18:

$$\frac{5x-2}{1+2x+3x^2}$$

Solution:

Let $5x - 2 = A\dfrac{d}{dx}\left(1 + 2x + 3x^2\right) + B$

$\Rightarrow 5x - 2 = A(2 + 6x) + B$

Equating the coefficients of x and constant term on both sides, we get

$5 = 6A \Rightarrow A = \dfrac{5}{6}$

$2A + B = -2 \Rightarrow B = -\dfrac{11}{3}$

$\therefore 5x - 2 = \dfrac{5}{6}(2 + 6x) + \left(-\dfrac{11}{3}\right)$

$\Rightarrow \displaystyle\int \dfrac{5x - 2}{1 + 2x + 3x^2}\,dx = \int \dfrac{\dfrac{5}{6}(2 + 6x) - \dfrac{11}{3}}{1 + 2x + 3x^2}\,dx$

$\Rightarrow \dfrac{5}{6}\displaystyle\int \dfrac{2 + 6x}{1 + 2x + 3x^2}\,dx - \dfrac{11}{3}\int \dfrac{1}{1 + 2x + 3x^2}\,dx$

Let

$I_1 = \displaystyle\int \dfrac{2 + 6x}{1 + 2x + 3x^2}\,dx$ and $I_2 = \displaystyle\int \dfrac{1}{1 + 2x + 3x^2}\,dx$

$\therefore \displaystyle\int \dfrac{5x - 2}{1 + 2x + 3x^2}\,dx = \dfrac{5}{6}I_1 - \dfrac{11}{3}I_2 \quad \ldots(1)$

$I_1 = \displaystyle\int \dfrac{2 + 6x}{1 + 2x + 3x^2}\,dx$

Put $1 + 2x + 3x^2 = t$

$\Rightarrow (2 + 6x)\,dx = dt$

$\therefore I_1 = \displaystyle\int \dfrac{dt}{t}$

$I_1 = \log|t|$

$I_1 = \log\left|1 + 2x + 3x^2\right| \quad \ldots(2)$

$I_2 = \displaystyle\int \dfrac{1}{1 + 2x + 3x^2}\,dx$

$1 + 2x + 3x^2$ can be written as $1 + 3\left(x^2 + \dfrac{2}{3}x\right)$

Thus,

$$1 + 3\left(x^2 + \frac{2}{3}x\right)$$

$$= 1 + 3\left(x^2 + \frac{2}{3}x + \frac{1}{9} - \frac{1}{9}\right)$$

$$= 1 + 3\left(x + \frac{1}{3}\right)^2 - \frac{1}{3}$$

$$= \frac{2}{3} + 3\left(x + \frac{1}{3}\right)^2$$

$$= 3\left[\left(x + \frac{1}{3}\right)^2 + \frac{2}{9}\right]$$

$$= 3\left[\left(x + \frac{1}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^2\right]$$

$$I_2 = \frac{1}{3}\int \frac{1}{\left[\left(x + \frac{1}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^2\right]} dx$$

$$= \frac{1}{3}\left[\frac{1}{\frac{\sqrt{2}}{3}} \tan^{-1}\left(\frac{x + \frac{1}{3}}{\frac{\sqrt{2}}{3}}\right)\right]$$

$$= \frac{1}{3}\left[\frac{3}{\sqrt{2}} \tan^{-1}\left(\frac{3x + 1}{\sqrt{2}}\right)\right]$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{3x + 1}{\sqrt{2}}\right) \quad \dots(3)$$

Substituting equations (2) and (3) in equation (1), we get

$$\int \frac{5x - 2}{1 + 2x + 3x^2} dx = \frac{5}{6}\left[\log\left|1 + 2x + 3x^2\right|\right] - \frac{11}{3}\left[\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{3x + 1}{\sqrt{2}}\right)\right] + C$$

$$= \frac{5}{6}\log\left|1 + 2x + 3x^2\right| - \frac{11}{3\sqrt{2}} \tan^{-1}\left(\frac{3x + 1}{\sqrt{2}}\right) + C$$

Question 19:

$$\frac{6x + 7}{\sqrt{(x - 5)(x - 4)}}$$

Solution:

$$\frac{6x + 7}{\sqrt{(x - 5)(x - 4)}} = \frac{6x + 7}{\sqrt{x^2 - 9x + 20}}$$

Put, $6x+7 = A\dfrac{d}{dx}\left(x^2 - 9x + 20\right) + B$

$\Rightarrow 6x + 7 = A(2x - 9) + B$

Equating the coefficients of x and constant term, we get

$2A = 6 \Rightarrow A = 3$

$-9A + B = 7 \Rightarrow B = 34$

$\therefore 6x + 7 = 3(2x - 9) + 34$

$\displaystyle\int \dfrac{6x+7}{\sqrt{x^2 - 9x + 20}} = \int \dfrac{3(2x-9)+34}{\sqrt{x^2 - 9x + 20}}\,dx$

$= 3\displaystyle\int \dfrac{(2x-9)}{\sqrt{x^2 - 9x + 20}}\,dx + 34\int \dfrac{1}{\sqrt{x^2 - 9x + 20}}\,dx$

Let $I_1 = \displaystyle\int \dfrac{2x-9}{\sqrt{x^2 - 9x + 20}}\,dx$ and $I_2 = \displaystyle\int \dfrac{1}{\sqrt{x^2 - 9x + 20}}\,dx$

$\therefore \displaystyle\int \dfrac{6x+7}{\sqrt{x^2 - 9x + 20}} = 3I_1 + 34I_2 \qquad \ldots(1)$

Then,

$I_1 = \displaystyle\int \dfrac{2x-9}{\sqrt{x^2 - 9x + 20}}\,dx$

Let $x^2 - 9x + 20 = t$

$\Rightarrow (2x - 9)\,dx = dt$

$\Rightarrow I_1 = \dfrac{dt}{\sqrt{t}}$

$I_1 = 2\sqrt{t}$

$I_1 = 2\sqrt{x^2 - 9x + 20} \qquad \ldots(2)$

and

$I_2 = \displaystyle\int \dfrac{1}{\sqrt{x^2 - 9x + 20}}\,dx$

$x^2 - 9x + 20 = x^2 - 9x + 20 + \dfrac{81}{4} - \dfrac{81}{4}$

Thus,

$$x^2 - 9x + 20 + \frac{81}{4} - \frac{81}{4}$$

$$= \left(x - \frac{9}{2}\right)^2 - \frac{1}{4}$$

$$= \left(x - \frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2$$

$$\Rightarrow I_2 = \int \frac{1}{\left(x - \frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2} dx$$

$$I_2 = \log \left| \left(x - \frac{9}{2}\right) + \sqrt{x^2 - 9x + 20} \right| \qquad \ldots(3)$$

Substituting equations (2) and (3) in (1), we get

$$\int \frac{6x + 7}{\sqrt{x^2 - 9x + 20}} dx = 3 \left[2\sqrt{x^2 - 9x + 20} \right] + 34 \log \left| \left(x - \frac{9}{2}\right) + \sqrt{x^2 - 9x + 20} \right| + C$$

$$= 6\sqrt{x^2 - 9x + 20} + 34 \log \left| \left(x - \frac{9}{2}\right) + \sqrt{x^2 - 9x + 20} \right| + C$$

Question 20:

$$\frac{x + 2}{\sqrt{4x - x^2}}$$

Solution:

Consider, $x + 2 = A \dfrac{d}{dx}(4x - x^2) + B$

$$\Rightarrow x + 2 = A(4 - 2x) + B$$

Equating the coefficients of x and constant term on both sides, we get

$$-2A = 1 \Rightarrow A = -\frac{1}{2}$$

$$4A + B = 2 \Rightarrow B = 4$$

$$\Rightarrow (x + 2) = -\frac{1}{2}(4 - 2x) + 4$$

$$\therefore \int \frac{x + 2}{\sqrt{4x - x^2}} dx = \int \frac{-\dfrac{1}{2}(4 - 2x) + 4}{\sqrt{(4x - x^2)}} dx$$

$$= -\frac{1}{2} \int \frac{(4 - 2x)}{\sqrt{(4x - x^2)}} dx + 4 \int \frac{1}{\sqrt{(4x - x^2)}} dx$$

Let $I_1 = \int \frac{4 - 2x}{\sqrt{4x - x^2}} dx$ and $I_2 = \int \frac{1}{\sqrt{4x - x^2}} dx$

$$\therefore \int \frac{x+2}{\sqrt{4x-x^2}} dx = -\frac{1}{2}I_1 + 4I_2 \qquad \ldots(1)$$

Then,

$$I_1 = \int \frac{4-2x}{\sqrt{4x-x^2}} dx$$

Let $4x - x^2 = t$

$$\Rightarrow (4-2x)dx = dt$$

$$\Rightarrow I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{4x-x^2} \qquad \ldots(2)$$

$$I_2 = \int \frac{1}{\sqrt{4x-x^2}} dx$$

$$\Rightarrow 4x - x^2 = -\left(-4x + x^2\right)$$

$$= \left(-4x + x^2 + 4 - 4\right)$$

$$= 4 - \left(x-2\right)^2$$

$$= \left(2\right)^2 - \left(x-2\right)^2$$

$$\therefore I_2 = \int \frac{1}{\sqrt{\left(2\right)^2 - \left(x-2\right)^2}} dx = \sin^{-1}\left(\frac{x-2}{2}\right) \quad \ldots(3)$$

Using equations (2) and (3) in (1), we get

$$\int \frac{x+2}{\sqrt{4x-x^2}} dx = -\frac{1}{2}\left(2\sqrt{4x-x^2}\right) + 4\sin^{-1}\left(\frac{x-2}{2}\right) + C$$

$$= -\sqrt{4x-x^2} + 4\sin^{-1}\left(\frac{x-2}{2}\right) + C$$

Question 21:

$$\frac{x+2}{\sqrt{x^2+2x+3}}$$

Solution:

$$\int \frac{x+2}{\sqrt{x^2+2x+3}} dx = \frac{1}{2}\int \frac{2(x+2)}{\sqrt{x^2+2x+3}} dx$$

$$= \frac{1}{2}\int \frac{2x+4}{\sqrt{x^2+2x+3}} dx$$

$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}} dx + \frac{1}{2}\int \frac{2}{\sqrt{x^2+2x+3}} dx$$

$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}} dx + \int \frac{1}{\sqrt{x^2+2x+3}} dx$$

Let $I_1 = \int \dfrac{2x+2}{\sqrt{x^2+2x+3}} dx$ and $I_2 = \int \dfrac{1}{\sqrt{x^2+2x+3}} dx$

$\therefore \int \dfrac{x+2}{\sqrt{x^2+2x+3}} dx = \dfrac{1}{2} I_1 + I_2$ ...(1)

Then, $I_1 = \int \dfrac{2x+2}{\sqrt{x^2+2x+3}} dx$

Put, $x^2+2x+3 = t$

$\Rightarrow (2x+2) dx = dt$

$I_1 = \int \dfrac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{x^2+2x+3}$ ...(2)

$I_2 = \int \dfrac{1}{\sqrt{x^2+2x+3}} dx$

$\Rightarrow x^2+2x+3 = x^2+2x+1+2 = (x+1)^2 + (\sqrt{2})^2$

$\therefore I_2 = \int \dfrac{1}{\sqrt{(x+1)^2 + (\sqrt{2})^2}} dx = \log\left|(x+1) + \sqrt{x^2+2x+3}\right|$ ...(3)

Using equations (2) and (3) in (1), we get

$\int \dfrac{x+2}{\sqrt{x^2+2x+3}} dx = \dfrac{1}{2}\left[2\sqrt{x^2+2x+3}\right] + \log\left|(x+1) + \sqrt{x^2+2x+3}\right| + C$

$= \sqrt{x^2+2x+3} + \log\left|(x+1) + \sqrt{x^2+2x+3}\right| + C$

Question 22:

$\dfrac{x+3}{x^2-2x-5}$

Solution:

Let $(x+3) = A\dfrac{d}{dx}(x^2-2x-5) + B$

$(x+3) = A(2x-2) + B$

Equating the coefficients of x and constant term on both sides, we get

$$2A = 1 \Rightarrow A = \frac{1}{2}$$

$$-2A + B = 3 \Rightarrow B = 4$$

$$\therefore (x + 3) = \frac{1}{2}(2x - 2) + 4$$

$$\Rightarrow \int \frac{x + 3}{x^2 - 2x - 5} dx = \int \frac{\frac{1}{2}(2x - 2) + 4}{x^2 - 2x - 5} dx$$

$$= \frac{1}{2} \int \frac{2x - 2}{x^2 - 2x - 5} dx + 4 \int \frac{1}{x^2 - 2x - 5} dx$$

Let $I_1 = \int \frac{2x - 2}{x^2 - 2x - 5} dx$ and $I_2 = \int \frac{1}{x^2 - 2x - 5} dx$

$$\therefore \int \frac{x + 3}{x^2 - 2x - 5} dx = \frac{1}{2} I_1 + 4 I_2 \quad \ldots(1)$$

Then, $I_1 = \int \frac{2x - 2}{x^2 - 2x - 5} dx$

Put, $x^2 - 2x - 5 = t$

$$\Rightarrow (2x - 2) dx = dt$$

$$\Rightarrow I_1 = \int \frac{dt}{t} = \log|t| = \log|x^2 - 2x - 5| \quad \ldots(2)$$

$$I_2 = \int \frac{1}{x^2 - 2x - 5} dx$$

$$= \int \frac{1}{(x^2 - 2x + 1) - 6} dx$$

$$= \int \frac{1}{(x - 1)^2 - (\sqrt{6})^2} dx$$

$$= \frac{1}{2\sqrt{6}} \log\left(\frac{x - 1 - \sqrt{6}}{x - 1 + \sqrt{6}}\right) \quad \ldots(3)$$

Substituting (2) and (3) in (1), we get

$$\int \frac{x + 3}{x^2 - 2x - 5} dx = \frac{1}{2} \log|x^2 - 2x - 5| + \frac{4}{2\sqrt{6}} \log\left|\frac{x - 1 - \sqrt{6}}{x - 1 + \sqrt{6}}\right| + C$$

$$= \frac{1}{2} \log|x^2 - 2x - 5| + \frac{2}{\sqrt{6}} \log\left|\frac{x - 1 - \sqrt{6}}{x - 1 + \sqrt{6}}\right| + C$$

Question 23:

$$\frac{5x + 3}{\sqrt{x^2 + 4x + 10}}$$

Solution:

Let $5x + 3 = A \dfrac{d}{dx}\left(x^2 + 4x + 10\right) + B$

$\Rightarrow 5x + 3 = A(2x + 4) + B$

Equating the coefficients of x and constant term, we get

$2A = 5 \Rightarrow A = \dfrac{5}{2}$

$4A + B = 3 \Rightarrow B = -7$

$\therefore 5x + 3 = \dfrac{5}{2}(2x + 4) - 7$

$\Rightarrow \displaystyle\int \dfrac{5x + 3}{\sqrt{x^2 + 4x + 10}}\, dx = \int \dfrac{\dfrac{5}{2}(2x + 4) - 7}{\sqrt{x^2 + 4x + 10}}\, dx$

$= \dfrac{5}{2}\displaystyle\int \dfrac{2x + 4}{\sqrt{x^2 + 4x + 10}}\, dx - 7\int \dfrac{1}{\sqrt{x^2 + 4x + 10}}\, dx$

Let $I_1 = \displaystyle\int \dfrac{2x + 4}{\sqrt{x^2 + 4x + 10}}\, dx$ and $I_2 = \int \dfrac{1}{\sqrt{x^2 + 4x + 10}}\, dx$

$\therefore \displaystyle\int \dfrac{5x + 3}{\sqrt{x^2 + 4x + 10}}\, dx = \dfrac{5}{2} I_1 - 7 I_2 \quad \ldots(1)$

Then,

$I_1 = \displaystyle\int \dfrac{2x + 4}{\sqrt{x^2 + 4x + 10}}\, dx$

Put, $x^2 + 4x + 10 = t$

$\therefore (2x + 4)\, dx = dt$

$\Rightarrow I_1 = \displaystyle\int \dfrac{dt}{t} = 2\sqrt{t} = 2\sqrt{x^2 + 4x + 10} \quad \ldots(2)$

$I_2 = \displaystyle\int \dfrac{1}{\sqrt{x^2 + 4x + 10}}\, dx$

$= \displaystyle\int \dfrac{1}{\sqrt{(x^2 + 4x + 4) + 6}}\, dx$

$= \displaystyle\int \dfrac{1}{\sqrt{(x + 2)^2 + \left(\sqrt{6}\right)^2}}\, dx$

$= \log\left|(x + 2)\sqrt{x^2 + 4x + 10}\right| \quad \ldots(3)$

Using equations (2) and (3) in (1), we get

$\displaystyle\int \dfrac{5x + 3}{\sqrt{x^2 + 4x + 10}}\, dx = \dfrac{5}{2}\left[2\sqrt{x^2 + 4x + 10}\right] - 7\log\left|(x + 2)\sqrt{x^2 + 4x + 10}\right| + C$

$= 5\sqrt{x^2 + 4x + 10} - 7\log\left|(x + 2)\sqrt{x^2 + 4x + 10}\right| + C$

Choose the correct answer in Exercises 24 and 25.

Question 24:

$\int \dfrac{dx}{x^2 + 2x + 2}$ equals

(A) $x \tan^{-1}(x+1) + C$

(B) $\tan^{-1}(x+1) + C$

(C) $(x+1)\tan^{-1} x + C$

(D) $\tan^{-1} x + C$

Solution:

$$\int \dfrac{dx}{x^2 + 2x + 2} = \int \dfrac{dx}{(x^2 + 2x + 1) + 1}$$

$$= \int \dfrac{1}{(x+1)^2 + (1)^2} dx$$
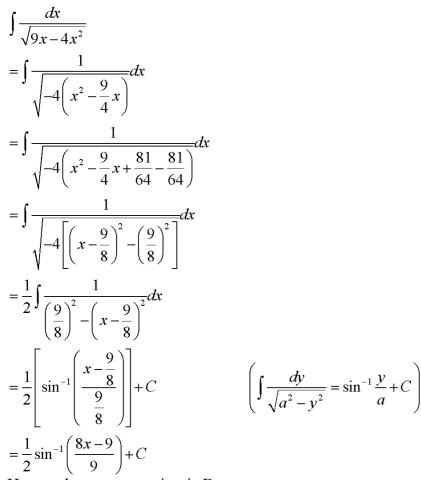
$$= \left[\tan^{-1}(x+1) \right] + C$$

Hence, the correct option is B.

Question 25:

$\int \dfrac{dx}{\sqrt{9x - 4x^2}}$ equals

(A) $\dfrac{1}{9} \sin^{-1}\left(\dfrac{9x - 8}{8} \right) + C$

(B) $\dfrac{1}{2} \sin^{-1}\left(\dfrac{8x - 9}{8} \right) + C$

(C) $\dfrac{1}{3} \sin^{-1}\left(\dfrac{9x - 8}{8} \right) + C$

(D) $\dfrac{1}{2} \sin^{-1}\left(\dfrac{9x - 8}{9} \right) + C$

Solution:

$$\int \frac{dx}{\sqrt{9x - 4x^2}}$$

$$= \int \frac{1}{\sqrt{-4\left(x^2 - \frac{9}{4}x\right)}} dx$$

$$= \int \frac{1}{\sqrt{-4\left(x^2 - \frac{9}{4}x + \frac{81}{64} - \frac{81}{64}\right)}} dx$$

$$= \int \frac{1}{\sqrt{-4\left[\left(x - \frac{9}{8}\right)^2 - \left(\frac{9}{8}\right)^2\right]}} dx$$

$$= \frac{1}{2} \int \frac{1}{\sqrt{\left(\frac{9}{8}\right)^2 - \left(x - \frac{9}{8}\right)^2}} dx$$

$$= \frac{1}{2}\left[\sin^{-1}\left(\frac{x - \frac{9}{8}}{\frac{9}{8}}\right)\right] + C \qquad \left(\int \frac{dy}{\sqrt{a^2 - y^2}} = \sin^{-1}\frac{y}{a} + C\right)$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{8x - 9}{9}\right) + C$$

Hence, the correct option is B.

EXERCISE 7.5

Integrate the rational functions in Exercises 1 to 21.

Question 1:

$$\frac{x}{(x+1)(x+2)}$$

Solution:

Let $\dfrac{x}{(x+1)(x+2)} = \dfrac{A}{(x+1)} + \dfrac{B}{(x+2)}$

$\Rightarrow x = A(x+2) + B(x+1)$

Equating the coefficients of x and constant term, we get

$A + B = 1$

$2A + B = 0$

On solving, we get

$A = -1$ and $B = 2$

$\therefore \dfrac{x}{(x+1)(x+2)} = \dfrac{-1}{(x+1)} + \dfrac{2}{(x+2)}$

$\Rightarrow \displaystyle\int \dfrac{x}{(x+1)(x+2)}\,dx = \int \dfrac{-1}{(x+1)} + \dfrac{2}{(x+2)}\,dx$

$= -\log|x+1| + 2\log|x+2| + C$

$= \log(x+2)^2 - \log(x+1) + C$

$= \log\dfrac{(x+2)^2}{(x+1)} + C$

Question 2:

$$\frac{1}{x^2 - 9}$$

Solution:

Let $\dfrac{1}{(x+3)(x-3)} = \dfrac{A}{(x+3)} + \dfrac{B}{(x-3)}$

$1 = A(x-3) + B(x+3)$

Equating the coefficients of x and constants term, we get

$A + B = 0$

$-3A + 3B = 1$

On solving, we get

$A = -\dfrac{1}{6}$ and $B = \dfrac{1}{6}$

$\therefore \dfrac{1}{(x+3)(x-3)} = \dfrac{-1}{6(x+3)} + \dfrac{1}{6(x-3)}$

$\Rightarrow \displaystyle\int \dfrac{1}{(x^2-9)} dx = \int \left(\dfrac{-1}{6(x+3)} + \dfrac{1}{6(x-3)} \right) dx$

$= -\dfrac{1}{6} \log|x+3| + \dfrac{1}{6} \log|x-3| + C$

$= \dfrac{1}{6} \log \dfrac{|(x-3)|}{|(x+3)|} + C$

Question 3:

$\dfrac{3x-1}{(x-1)(x-2)(x-3)}$

Solution:

Let $\dfrac{3x-1}{(x-1)(x-2)(x-3)} = \dfrac{A}{(x-1)} + \dfrac{B}{(x-2)} + \dfrac{C}{(x-3)}$

$3x-1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$ ...(1)

Equating the coefficients of x^2, x and constant terms, we get

$A + B + C = 0$

$-5A - 4B - 3C = 3$

$6A + 3B + 2C = -1$

Solving these equations, we get

$A = 1, B = -5$ and $C = 4$

$\therefore \dfrac{3x-1}{(x-1)(x-2)(x-3)} = \dfrac{1}{(x-1)} - \dfrac{5}{(x-2)} + \dfrac{4}{(x-3)}$

$\Rightarrow \displaystyle\int \dfrac{3x-1}{(x-1)(x-2)(x-3)} dx = \int \left\{ \dfrac{1}{(x-1)} - \dfrac{5}{(x-2)} + \dfrac{4}{(x-3)} \right\} dx$

$= \log|x-1| - 5\log|x-2| + 4\log|x-3| + C$

Question 4:

$\dfrac{x}{(x-1)(x-2)(x-3)}$

Solution:

Let $\dfrac{x}{(x-1)(x-2)(x-3)} = \dfrac{A}{(x-1)} + \dfrac{B}{(x-2)} + \dfrac{C}{(x-3)}$

$x = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$ ...(1)

Equating the coefficients of x^2, x and constant terms, we get

$A + B + C = 0$

$-5A - 4B - 3C = 1$

$6A + 4B + 2C = 0$

Solving these equations, we get

$A = \dfrac{1}{2}, B = -2$ and $C = \dfrac{3}{2}$

$\therefore \dfrac{x}{(x-1)(x-2)(x-3)} = \dfrac{1}{2(x-1)} - \dfrac{2}{(x-2)} + \dfrac{3}{2(x-3)}$

$\Rightarrow \int \dfrac{x}{(x-1)(x-2)(x-3)} dx = \int \left\{ \dfrac{1}{2(x-1)} - \dfrac{2}{(x-2)} + \dfrac{3}{2(x-3)} \right\} dx$

$= \dfrac{1}{2} \log|x-1| - 2\log|x-2| + \dfrac{3}{2}\log|x-3| + C$

Question 5:

$\dfrac{2x}{x^2 + 3x + 2}$

Solution:

Let $\dfrac{2x}{x^2 + 3x + 2} = \dfrac{A}{(x+1)} + \dfrac{B}{(x+2)}$

$2x = A(x+2) + B(x+1)$ ...(1)

Equating the coefficients of x and constant terms, we get

$A + B = 2$

$2A + B = 0$

Solving these equations, we get

$A = -2$ and $B = 4$

$\therefore \dfrac{2x}{(x+1)(x+2)} = \dfrac{-2}{(x+1)} + \dfrac{4}{(x+2)}$

$\Rightarrow \int \dfrac{2x}{(x+1)(x+2)} dx = \int \left\{ \dfrac{4}{(x+2)} - \dfrac{2}{(x+1)} \right\} dx$

$= 4\log|x+2| - 2\log|x+1| + C$

Question 6:

$\dfrac{1 - x^2}{x(1 - 2x)}$

Solution:

It can be seen that the given integrand is not a proper fraction.

Therefore, on dividing $(1 - x^2)$ by $x(1 - 2x)$, we get

$$\frac{1-x^2}{x(1-2x)} = \frac{1}{2} + \frac{1}{2}\left(\frac{2-x}{x(1-2x)}\right) \quad \ldots(1)$$

Let $\dfrac{2-x}{x(1-2x)} = \dfrac{A}{x} + \dfrac{B}{(1-2x)}$

$$\Rightarrow (2-x) = A(1-2x) + Bx$$

Equating the coefficients of x and constant term, we get

$$-2A + B = -1$$

And, $A = 2$

Solving these equations, we get

$A = 2$ and $B = 3$

$$\therefore \frac{2-x}{x(1-2x)} = \frac{2}{x} + \frac{3}{(1-2x)}$$

Substituting in equation (1), we get

$$\frac{1-x^2}{x(1-2x)} = \frac{1}{2} + \frac{1}{2}\left\{\frac{2}{x} + \frac{3}{(1-2x)}\right\}$$

$$\Rightarrow \int \frac{1-x^2}{x(1-2x)} dx = \int \left\{\frac{1}{2} + \frac{1}{2}\left(\frac{2}{x} + \frac{3}{(1-2x)}\right)\right\} dx$$

$$= \frac{x}{2} + \log|x| + \frac{3}{2(-2)}\log|1-2x| + C$$

$$= \frac{x}{2} + \log|x| - \frac{3}{4}\log|1-2x| + C$$

Question 7:

$$\frac{x}{(x^2+1)(x-1)}$$

Solution:

Let $\dfrac{x}{(x^2+1)(x-1)} = \dfrac{Ax+B}{(x^2+1)} + \dfrac{C}{(x-1)}$ $\quad$ ......(1)

$$x = (Ax+B)(x-1) + C(x^2+1)$$

$$x = Ax^2 - Ax + Bx - B + Cx^2 + C$$

Equating the coefficients of x^2, x and constant term, we get

$$A + C = 0$$

$$-A + B = 1$$

$$-B + C = 0$$

On solving these equations, we get

$$A = -\frac{1}{2}, B = \frac{1}{2} \text{ and } C = \frac{1}{2}$$

From equation (1), we get

$$\therefore \frac{x}{\left(x^2+1\right)\left(x-1\right)} = \frac{\left(-\frac{1}{2}x+\frac{1}{2}\right)}{x^2+1} + \frac{\frac{1}{2}}{\left(x-1\right)}$$

$$\Rightarrow \int \frac{x}{\left(x^2+1\right)\left(x-1\right)} = -\frac{1}{2}\int \frac{x}{x^2+1}dx + \frac{1}{2}\int \frac{1}{x^2+1}dx + \frac{1}{2}\int \frac{1}{x-1}dx$$

$$= -\frac{1}{4}\int \frac{2x}{x^2+1}dx + \frac{1}{2}\tan^{-1}x + \frac{1}{2}\log|x-1|+C$$

Consider $\int \frac{2x}{x^2+1}dx$, let $\left(x^2+1\right)=t \Rightarrow 2xdx=dt$

$$\Rightarrow \int \frac{2x}{x^2+1}dx = \int \frac{dt}{t} = \log|t| = \log|x^2+1|$$

$$\therefore \int \frac{x}{\left(x^2+1\right)\left(x-1\right)} = -\frac{1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1}x + \frac{1}{2}\log|x-1|+C$$

$$= \frac{1}{2}\log|x-1| - \frac{1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1}x+C$$

Question 8:

$$\frac{x}{\left(x-1\right)^2\left(x+2\right)}$$

Solution:

Let $\dfrac{x}{\left(x-1\right)^2\left(x+2\right)} = \dfrac{A}{\left(x-1\right)} + \dfrac{B}{\left(x-1\right)^2} + \dfrac{C}{\left(x+2\right)}$

$$x = A\left(x-1\right)\left(x+2\right) + B\left(x+2\right) + C\left(x-1\right)^2$$

Equating the coefficients of x^2, x and constant term, we get

$A+C=0$

$A+B-2C=1$

$-2A+2B+C=0$

On solving these equations, we get

$A=\dfrac{2}{9}, B=\dfrac{1}{3}$ and $C=-\dfrac{2}{9}$

$$\therefore \frac{x}{\left(x-1\right)^2\left(x+2\right)} = \frac{2}{9\left(x+1\right)} + \frac{1}{3\left(x-1\right)^2} - \frac{2}{9\left(x+2\right)}$$

$$\Rightarrow \int \frac{x}{\left(x-1\right)^2\left(x+2\right)}dx = \frac{2}{9}\int \frac{1}{\left(x-1\right)}dx + \frac{1}{3}\int \frac{1}{\left(x-1\right)^2}dx - \frac{2}{9}\int \frac{1}{\left(x-2\right)}dx$$

$$= \frac{2}{9}\log|x-1| + \frac{1}{3}\left(\frac{-1}{x-1}\right) - \frac{2}{9}\log|x+2|+C$$

$$= \frac{2}{9}\log\left|\frac{x-1}{x+2}\right| - \frac{1}{3\left(x-1\right)}+C$$

Question 9:

$$\frac{3x+5}{x^3-x^2-x+1}$$

Solution:

$$\frac{3x+5}{x^3-x^2-x+1}=\frac{3x+5}{(x-1)^2(x+1)}$$

Let $\dfrac{3x+5}{(x-1)^2(x+1)}=\dfrac{A}{(x-1)}+\dfrac{B}{(x-1)^2}+\dfrac{C}{(x+1)}$

$3x+5=A(x-1)(x+1)+B(x+1)+C(x-1)^2$

$3x+5=A(x-1)(x+1)+B(x+1)+C(x^2-2x+1)$ ...(1)

Equating the coefficients of x^2, x and constant term, we get

$A+C=0$

$B-2C=3$

$-A+B+C=5$

On solving these equations, we get

$A=-\dfrac{1}{2}, B=4$ and $C=\dfrac{1}{2}$

$\therefore \dfrac{3x+5}{(x-1)^2(x+1)}=\dfrac{-1}{2(x-1)}+\dfrac{4}{(x-1)^2}+\dfrac{1}{2(x+1)}$

$\Rightarrow \displaystyle\int\frac{3x+5}{(x-1)^2(x+1)}dx=-\frac{1}{2}\int\frac{1}{(x-1)}dx+4\int\frac{1}{(x-1)^2}dx+\frac{1}{2}\int\frac{1}{(x+1)}dx$

$=-\dfrac{1}{2}\log|x-1|+4\left(\dfrac{-1}{x-1}\right)+\dfrac{1}{2}\log|x+1|+C$

$=\dfrac{1}{2}\log\left|\dfrac{x+1}{x-1}\right|-\dfrac{4}{(x-1)}+C$

Question 10:

$$\frac{2x-3}{(x^2-1)(2x+3)}$$

Solution:

$$\frac{2x-3}{(x^2-1)(2x+3)}=\frac{2x-3}{(x-1)(x+1)(2x+3)}$$

Let $\dfrac{2x-3}{(x-1)(x+1)(2x+3)}=\dfrac{A}{(x+1)}+\dfrac{B}{(x-1)}+\dfrac{C}{(2x+3)}$

$$\Rightarrow (2x-3) = A(x-1)(2x-3) + B(x+1)(2x+3) + C(x+1)(x-1)$$
$$\Rightarrow (2x-3) = A(2x^2+x-3) + B(2x^2+5x+3) + C(x^2-1)$$
$$\Rightarrow (2x-3) = (2A+2B+C)x^2 + (A+5B)x + (-3A+3B-C)$$

Equating the coefficients of x^2, x and constant term, we get

$$2A+2B+C = 0$$
$$A+5B = 2$$
$$-3A+3B-C = -3$$

On solving, we get

$$A = \frac{5}{2}, B = -\frac{1}{10} \text{ and } C = -\frac{24}{5}$$

$$\therefore \frac{2x-3}{(x+1)(x-1)(2x+3)} = \frac{5}{2(x+1)} - \frac{1}{10(x-1)} - \frac{24}{5(2x+3)}$$

$$\Rightarrow \int \frac{2x-3}{(x+1)(x-1)(x+1)} dx = \frac{5}{2} \int \frac{1}{(x+1)} dx - \frac{1}{10} \int \frac{1}{(x-1)} dx - \frac{24}{5} \int \frac{1}{(2x+3)} dx$$

$$= \frac{5}{2} \log|x+1| - \frac{1}{10} \log|x-1| - \frac{24}{5 \times 2} \log|2x+3| + C$$

$$= \frac{5}{2} \log|x+1| - \frac{1}{10} \log|x-1| - \frac{12}{5} \log|2x+3| + C$$

Question 11:

$$\frac{5x}{(x+1)(x^2-4)}$$

Solution:

$$\frac{5x}{(x+1)(x^2-4)} = \frac{5x}{(x+1)(x+2)(x-2)}$$

Let $\dfrac{5x}{(x+1)(x^2-4)} = \dfrac{A}{(x+1)} + \dfrac{B}{(x+2)} + \dfrac{C}{(x-2)}$

$$5x = A(x+2)(x-2) + B(x+1)(x-2) + C(x+1)(x+2) \quad \ldots(1)$$

Equating the coefficients of x^2, x and constant term, we get

$$A+B+C = 0$$
$$-B+3C = 5$$
$$-4A-2B+2C = 0$$

On solving, we get

$A = \dfrac{5}{3}, B = -\dfrac{5}{2}$ and $C = \dfrac{5}{6}$

$\therefore \dfrac{5x}{(x+1)(x+2)(x-2)} = \dfrac{5}{3(x+1)} - \dfrac{5}{2(x+2)} + \dfrac{5}{6(x-2)}$

$\Rightarrow \displaystyle\int \dfrac{5x}{(x+1)(x+2)(x-2)}\, dx = \dfrac{5}{3}\displaystyle\int \dfrac{1}{(x+1)}\, dx - \dfrac{5}{2}\displaystyle\int \dfrac{1}{(x+2)}\, dx + \dfrac{5}{6}\displaystyle\int \dfrac{1}{(x-2)}\, dx$

$= \dfrac{5}{3}\log|x+1| - \dfrac{5}{2}\log|x+2| + \dfrac{5}{6}\log|x-2| + C$

Question 12:

$\dfrac{x^3 + x + 1}{x^2 - 1}$

Solution:

On dividing $\left(x^3 + x + 1\right)$ by $x^2 - 1$, we get

$\dfrac{x^3 + x + 1}{x^2 - 1} = x + \dfrac{2x+1}{x^2 - 1}$

Let $\dfrac{2x+1}{x^2 - 1} = \dfrac{A}{(x+1)} + \dfrac{B}{(x+1)}$

$2x + 1 = A(x-1) + B(x+1)$ ...(1)

Equating the coefficients of x and constant term, we get

$A + B = 2$

$-A + B = 1$

On solving, we get

$A = \dfrac{1}{2}$ and $B = \dfrac{3}{2}$

$\therefore \dfrac{x^3 + x + 1}{x^2 - 1} = x + \dfrac{1}{2(x+1)} + \dfrac{3}{2(x-1)}$

$\Rightarrow \displaystyle\int \dfrac{x^3 + x + 1}{x^2 - 1}\, dx = \displaystyle\int x\, dx + \dfrac{1}{2}\displaystyle\int \dfrac{1}{(x+1)}\, dx + \dfrac{3}{2}\displaystyle\int \dfrac{1}{(x-1)}\, dx$

$= \dfrac{x^2}{2} + \dfrac{1}{2}\log|x+1| + \dfrac{3}{2}\log|x-1| + C$

Question 13:

$\dfrac{2}{(1-x)(1+x^2)}$

Solution:

Let $\dfrac{2}{(1-x)(1+x^2)} = \dfrac{A}{(1-x)} + \dfrac{Bx+C}{(1+x^2)}$

$2 = A(1+x^2) + (Bx+C)(1-x)$

$2 = A + Ax^2 + Bx - Bx^2 + C - Cx$

Equating the coefficients of x^2, x and constant term, we get

$A - B = 0$

$B - C = 0$

$A + C = 2$

On solving these equations, we get

$A = 1, B = 1$ and $C = 1$

$\therefore \dfrac{2}{(1-x)(1+x^2)} = \dfrac{1}{1-x} + \dfrac{x+1}{1+x^2}$

$\Rightarrow \displaystyle\int \dfrac{2}{(1-x)(1+x^2)}\,dx = \int \dfrac{1}{1-x}\,dx + \int \dfrac{x}{1+x^2}\,dx + \int \dfrac{1}{1+x^2}\,dx$

$= -\displaystyle\int \dfrac{1}{1-x}\,dx + \dfrac{1}{2}\int \dfrac{2x}{1+x^2}\,dx + \int \dfrac{1}{1+x^2}\,dx$

$= -\log|x-1| + \dfrac{1}{2}\log|1+x^2| + \tan^{-1}x + C$

Question 14:

$\dfrac{3x-1}{(x+2)^2}$

Solution:

Let $\dfrac{3x-1}{(x+2)^2} = \dfrac{A}{(x+2)} + \dfrac{B}{(x+2)^2}$

$\Rightarrow 3x-1 = A(x+2) + B$

Equating the coefficient of x and constant term, we get

$A = 3$

$2A + B = -1 \Rightarrow B = -7$

$\therefore \dfrac{3x-1}{(x+2)^2} = \dfrac{3}{(x+2)} - \dfrac{7}{(x+2)^2}$

$\Rightarrow \displaystyle\int \dfrac{3x-1}{(x+2)^2}dx = 3\int \dfrac{1}{(x+2)}dx - 7\int \dfrac{x}{(x+2)^2}dx$

$= 3\log|x+2| - 7\left(\dfrac{-1}{(x+2)}\right) + C$

$= 3\log|x+2| + \dfrac{7}{(x+2)} + C$

Question 15:

$\dfrac{1}{x^4 - 1}$

Solution:

$\dfrac{1}{(x^4-1)} = \dfrac{1}{(x^2-1)(x^2+1)} = \dfrac{1}{(x+1)(x-1)(x^2+1)}$

Let $\dfrac{1}{(x+1)(x-1)(x^2+1)} = \dfrac{A}{(x+1)} + \dfrac{B}{(x-1)} + \dfrac{Cx+D}{(x^2+1)}$

$1 = A(x-1)(1+x^2) + B(x+1)(1+x^2) + (Cx+D)(x^2-1)$

$1 = A(x^3 + x - x^2 - 1) + B(x^3 + x + x^2 + 1) + Cx^3 + Dx^2 - Cx - D$

$1 = (A+B+C)x^3 + (-A+B+D)x^2 + (A+B-C)x + (-A+B-D)$

Equating the coefficients of x^3, x^2, x and constant term, we get

$A + B + C = 0$

$-A + B + D = 0$

$A + B - C = 0$

$-A + B - D = 1$

On solving, we get

$A = \dfrac{-1}{4}, B = \dfrac{1}{4}, C = 0$ and $D = -\dfrac{1}{2}$

$\therefore \dfrac{1}{(x^4 - 1)} = \dfrac{-1}{4(x+1)} + \dfrac{1}{4(x-1)} + \dfrac{1}{2(1+x^2)}$

$\Rightarrow \displaystyle\int \dfrac{1}{(x^4 - 1)} dx = \int \dfrac{-1}{4(x+1)} dx + \int \dfrac{1}{4(x-1)} dx - \int \dfrac{1}{2(1+x^2)} dx$

$\Rightarrow \displaystyle\int \dfrac{1}{(x^4 - 1)} dx = -\dfrac{1}{4}\log|x+1| + \dfrac{1}{4}\log|x-1| - \dfrac{1}{2}\tan^{-1}x + C$

$= \dfrac{1}{4}\log\left|\dfrac{x-1}{x+1}\right| - \dfrac{1}{2}\tan^{-1}x + C$

Question 16:

$\dfrac{1}{x(x^n + 1)}$

[Hint: multiply numerator and denominator by x^{n-1} and put $x^n = t$]

Solution:

$\dfrac{1}{x(x^n + 1)}$

Multiplying numerator and denominator by x^{n-1}, we get

$\dfrac{1}{x(x^n + 1)} = \dfrac{x^{n-1}}{x^{n-1}x(x^n + 1)} = \dfrac{x^{n-1}}{x^n(x^n + 1)}$

Let $x^n = t \Rightarrow nx^{n-1}dx = dt$

$\therefore \displaystyle\int \dfrac{1}{x(x^n + 1)} dx = \int \dfrac{x^{n-1}}{x^n(x^n + 1)} dx = \dfrac{1}{n}\int \dfrac{1}{t(t+1)} dt$

Let $\dfrac{1}{t(t+1)} = \dfrac{A}{t} + \dfrac{B}{(t+1)}$

$1 = A(1+t) + Bt$ ...(1)

Equating the coefficients of t and constant term, we get

$A = 1$ and $B = -1$

$$\therefore \frac{1}{t(t+1)} = \frac{1}{t} - \frac{1}{(1+t)}$$

$$\Rightarrow \int \frac{1}{x(x^n+1)} dx = \frac{1}{n} \int \left\{ \frac{1}{t} - \frac{1}{(1+t)} \right\} dx$$

$$= \frac{1}{n} \left[\log|t| - \log|t+1| \right] + C$$

$$= \frac{1}{n} \left[\log|x^n| - \log|x^n+1| \right] + C$$

$$= \frac{1}{n} \log \left| \frac{x^n}{x^n+1} \right| + C$$

Question 17:

$$\frac{\cos x}{(1-\sin x)(2-\sin x)}$$ [Hint: Put $\sin x = t$]

Solution:

$$\frac{\cos x}{(1-\sin x)(2-\sin x)}$$ Put, $\sin x = t \Rightarrow \cos x\, dx = dt$

$$\therefore \int \frac{\cos x}{(1-\sin x)(2-\sin x)} dx = \int \frac{dt}{(1-t)(2-t)}$$

Let $$\frac{1}{(1-t)(2-t)} = \frac{A}{(1-t)} + \frac{B}{(2-t)}$$

$$1 = A(2-t) + B(1-t) \quad \dots(1)$$

Equating the coefficients of t and constant, we get

$-2A - B = 0$, and $2A + B = 1$

On solving, we get

$A = 1$ and $B = -1$

$$\therefore \frac{1}{(1-t)(2-t)} = \frac{1}{(1-t)} - \frac{1}{(2-t)}$$

$$\Rightarrow \int \frac{\cos x}{(1-\sin x)(2-\sin x)} dx = \int \left\{ \frac{1}{1-t} - \frac{1}{(2-t)} \right\} dt$$

$$= -\log|1-t| + \log|2-t| + C$$

$$= \log \left| \frac{2-t}{1-t} \right| + C$$

$$= \log \left| \frac{2-\sin x}{1-\sin x} \right| + C$$

Question 18:

$$\frac{\left(x^2+1\right)\left(x^2+2\right)}{\left(x^2+3\right)\left(x^2+4\right)}$$

Solution:

$$\frac{\left(x^2+1\right)\left(x^2+2\right)}{\left(x^2+3\right)\left(x^2+4\right)} = \frac{\left(4x^2+10\right)}{\left(x^2+3\right)\left(x^2+4\right)}$$

Let $\frac{\left(4x^2+10\right)}{\left(x^2+3\right)\left(x^2+4\right)} = \frac{Ax+B}{\left(x^2+3\right)} + \frac{Cx+D}{\left(x^2+4\right)}$

$4x^2+10 = \left(Ax+B\right)\left(x^2+4\right) + \left(Cx+D\right)\left(x^2+3\right)$

$4x^2+10 = Ax^3 + 4Ax + Bx^2 + 4B + Cx^3 + 3Cx + Dx^2 + 3D$

$4x^2+10 = \left(A+C\right)x^3 + \left(B+D\right)x^2 + \left(4A+3C\right)x + \left(4B+3D\right)$

Equating the coefficients of x^3, x^2, x and constant term, we get

$A+C=0$

$B+D=4$

$4A+3C=0$

$4B+3D=10$

On solving these equations, we get

$A=0, B=-2, C=0$ and $D=6$

$$\therefore \frac{\left(4x^2+10\right)}{\left(x^2+3\right)\left(x^2+4\right)} = \frac{-2}{\left(x^2+3\right)} + \frac{6}{\left(x^2+4\right)}$$

$$\frac{\left(x^2+1\right)\left(x^2+2\right)}{\left(x^2+3\right)\left(x^2+4\right)} = \left(\frac{-2}{\left(x^2+3\right)} + \frac{6}{\left(x^2+4\right)}\right)$$

$$\Rightarrow \int \frac{\left(x^2+1\right)\left(x^2+2\right)}{\left(x^2+3\right)\left(x^2+4\right)} dx = \int 1 - \left\{\frac{-2}{\left(x^2+3\right)} + \frac{6}{\left(x^2+4\right)}\right\} dx$$

$$= \int \left\{1 + \frac{2}{x^2 + \left(\sqrt{3}\right)^2} - \frac{6}{x^2 + 2^2}\right\} dx$$

$$= x + 2\left(\frac{1}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}}\right) - 6\left(\frac{1}{2} \tan^{-1} \frac{x}{2}\right) + C$$

$$= x + \frac{2}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}} - 3\tan^{-1} \frac{x}{2} + C$$

Question 19:

$$\frac{2x}{\left(x^2+1\right)\left(x^2+3\right)}$$

Solution:

$$\frac{2x}{\left(x^2+1\right)\left(x^2+3\right)}$$

Put, $x^2 = t \Rightarrow 2x\,dx = dt$

$$\therefore \int \frac{2x}{\left(x^2+1\right)\left(x^2+3\right)}\,dx = \int \frac{dt}{(t+1)(t+3)} \quad ...(1)$$

Let $\dfrac{1}{(t+1)(t+3)} = \dfrac{A}{(t+1)} + \dfrac{B}{(t+3)}$

$1 = A(t+3) + B(t+1) \quad ...(2)$

Equating the coefficients of t and constant, we get

$A + B = 0$ and $3A + B = 1$

On solving, we get

$$A = \frac{1}{2} \text{ and } B = -\frac{1}{2}$$

$$\therefore \frac{1}{(t+1)(t+3)} = \frac{1}{2(t+1)} + \frac{1}{2(t+3)}$$

$$\Rightarrow \int \frac{2x}{\left(x^2+1\right)\left(x^2+3\right)}\,dx = \int \left\{ \frac{1}{2(t+1)} - \frac{1}{2(t+3)} \right\} dt$$

$$= \frac{1}{2}\log\left|(t+1)\right| - \frac{1}{2}\log\left|t+3\right| + C$$

$$= \frac{1}{2}\log\left|\frac{t+1}{t+3}\right| + C = \frac{1}{2}\log\left|\frac{x^2+1}{x^2+3}\right| + C$$

Question 20:

$$\frac{1}{x\left(x^4-1\right)}$$

Solution:

$$\frac{1}{x\left(x^4-1\right)}$$

Multiplying Nr and Dr by x^3, we get

$$\frac{1}{x(x^4-1)} = \frac{x^3}{x^4(x^4-1)}$$

$$\therefore \int \frac{1}{x(x^4-1)} dx = \int \frac{x^3}{x^4(x^4-1)} dx$$

Put, $x^4 = t \Rightarrow 4x^3 = dt$

$$\therefore \int \frac{1}{x(x^4-1)} dx = \frac{1}{4} \int \frac{dt}{t(t-1)}$$

Let $\dfrac{1}{t(t-1)} = \dfrac{A}{t} + \dfrac{B}{(t-1)}$

$1 = A(t-1) + Bt$ ...(1)

Equating the coefficients of t and constant, we get

$A = -1$ and $B = 1$

$$\Rightarrow \frac{1}{t(t-1)} = \frac{-1}{t} + \frac{1}{t-1}$$

$$\Rightarrow \int \frac{1}{x(x^4-1)} dx = \frac{1}{4} \int \left\{ \frac{-1}{t} + \frac{1}{t-1} \right\} dt$$

$$= \frac{1}{4} \left[-\log|t| + \log|t-1| \right] + C$$

$$= \frac{1}{4} \log \left| \frac{t-1}{t} \right| + C = \frac{1}{4} \log \left| \frac{x^4-1}{x^4} \right| + C$$

Question 21:

$\dfrac{1}{(e^x-1)}$ [Hint: Put $e^x = t$]

Solution:

Put $e^x = t \Rightarrow e^x dx = dt$

$$\Rightarrow \int \frac{1}{(e^x-1)} dx = \int \frac{1}{t-1} \times \frac{dt}{t} = \int \frac{1}{t(t-1)} dt$$

Let $\dfrac{1}{t(t-1)} = \dfrac{A}{t} + \dfrac{B}{t-1}$

$1 = A(t-1) + Bt$ ...(1)

Equating the coefficients of t and constant, we get

$A = -1$ and $B = 1$

$$\therefore \frac{1}{t(t-1)} = \frac{-1}{t} + \frac{1}{t-1}$$

$$\Rightarrow \int \frac{1}{t(t-1)} dt = \log\left|\frac{t-1}{t}\right| + C$$

$$= \log\left|\frac{e^x - 1}{e^x}\right| + C$$

Question 22:

$\int \dfrac{x\,dx}{(x-1)(x-2)}$ equals

A. $\log\left|\dfrac{(x-1)^2}{(x-2)}\right| + C$

B. $\log\left|\dfrac{(x-2)^2}{(x-1)}\right| + C$

C. $\log\left|\left(\dfrac{x-1}{x-2}\right)^2\right| + C$

D. $\log|(x-1)(x-2)| + C$

Solution:

Let $\dfrac{x}{(x-1)(x-2)} = \dfrac{A}{(x-1)} + \dfrac{B}{(x-2)}$

$x = A(x-2) + B(x-1)$　　...(1)

Equating the coefficients of x and constant, we get

$A = -1$ and $B = 2$

$$\therefore \frac{x}{(x-1)(x-2)} = \frac{-1}{(x-1)} + \frac{2}{(x-2)}$$

$$\Rightarrow \int \frac{x}{(x-1)(x-2)} dx = \int \left\{ \frac{-1}{(x-1)} + \frac{2}{(x-2)} \right\} dx$$

$$= -\log|x-1| + 2\log|x-2| + C$$

$$= \log\left|\frac{(x-2)^2}{x-1}\right| + C$$

Thus, the correct option is *B*.

Question 23:

$\int \dfrac{dx}{x\left(x^2+1\right)}$ equals

A. $\log|x| - \dfrac{1}{2}\log\left(x^2+1\right)+C$

B. $\log|x| + \dfrac{1}{2}\log\left(x^2+1\right)+C$

C. $-\log|x| + \dfrac{1}{2}\log\left(x^2+1\right)+C$

D. $\dfrac{1}{2}\log|x| + \log\left(x^2+1\right)+C$

Solution:

Let $\dfrac{1}{x\left(x^2+1\right)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$

$1 = A\left(x^2+1\right)+(Bx+C)x$

Equating the coefficients of x^2, x and constant terms, we get

$A+B=0$

$C=0$

$A=1$

On solving these equations, we get

$A=1$ $B=-1$ and $C=0$

$\therefore \dfrac{1}{x\left(x^2+1\right)} = \dfrac{1}{x} + \dfrac{-x}{x^2+1}$

$\Rightarrow \int \dfrac{1}{x\left(x^2+1\right)}\,dx = \int \left\{ \dfrac{1}{x} - \dfrac{x}{x^2-1}\right\}dx$

$= \log|x| - \dfrac{1}{2}\log\left|x^2+1\right|+C$

Thus, the correct option is A.

EXERCISE 7.6

Integrate the functions in Exercises 1 to 22.

Question 1:

$x \sin x$

Solution:

Let $I = \int x \sin x \, dx$

Taking $u = x$ and $v = \sin x$ and integrating by parts,

$$I = x \int \sin x \, dx - \int \left\{ \left(\frac{d}{dx}(x) \right) \int \sin x \, dx \right\} dx$$

$$= x(-\cos x) - \int 1 \cdot (-\cos x) \, dx$$

$$= -x \cos x + \sin x + C$$

Question 2:

$x \sin 3x$

Solution:

Let $I = \int x \sin 3x \, dx$

Taking $u = x$ and $v = \sin 3x$ and integrating by parts,

$$I = x \int \sin 3x \, dx - \int \left\{ \left(\frac{d}{dx} x \right) \int \sin 3x \, dx \right\} dx$$

$$= x \left(\frac{-\cos 3x}{3} \right) - \int 1 \cdot \left(\frac{-\cos 3x}{3} \right) dx$$

$$= \frac{-x \cos 3x}{3} + \frac{1}{3} \int \cos 3x \, dx$$

$$= \frac{-x \cos 3x}{3} + \frac{1}{9} \sin 3x + C$$

Question 3:

$x^2 e^x$

Solution:

Let $I = \int x^2 e^x \, dx$

Taking $u = x^2$ and $v = e^x$ and integrating by parts, we get

$$I = x^2 \int e^x dx - \int \left\{ \left(\frac{d}{dx} x^2 \right) \int e^x dx \right\} dx$$

$$= x^2 e^x - \int 2x.e^x dx$$

$$= x^2 e^x - 2\int x.e^x dx$$

Again using integration by parts, we get

$$= x^2 e^x - 2\left[x\int e^x dx - \int \left\{ \left(\frac{d}{dx} x \right) \int e^x dx \right\} dx \right]$$

$$= x^2 e^x - 2\left[xe^x - \int e^x dx \right]$$

$$= x^2 e^x - 2\left[xe^x - e^x \right]$$

$$= x^2 e^x - 2xe^x + 2e^x + C$$

$$= e^x \left(x^2 - 2x + 2 \right) + C$$

Question 4:

$x \log x$

Solution:

Let $I = \int x \log x \, dx$

Taking $u = \log x$ and $v = x$ and integrating by parts, we get

$$I = \log x \int x \, dx - \int \left\{ \left(\frac{d}{dx} \log x \right) \int x \, dx \right\} dx$$

$$= \log x . \frac{x^2}{2} - \int \frac{1}{x} . \frac{x^2}{2} dx$$

$$= \frac{x^2 \log x}{2} - \int \frac{x}{2} dx$$

$$= \frac{x^2 \log x}{2} - \frac{x^2}{4} + C$$

Question 5:

$x \log 2x$

Solution:

Let $I = \int x \log 2x \, dx$

Taking $u = \log 2x$ and $v = x$ and integrating by parts, we get

$$I = \log 2x \int x \, dx - \int \left\{ \left(\frac{d}{dx} \log 2x \right) \int x \, dx \right\} dx$$

$$= \log 2x . \frac{x^2}{2} - \int \frac{2}{2x} . \frac{x^2}{2} dx$$

$$= \frac{x^2 \log 2x}{2} - \int \frac{x}{2} dx$$

$$= \frac{x^2 \log 2x}{2} - \frac{x^2}{4} + C$$

Question 6:

$x^2 \log x$

Solution:

Let $I = \int x^2 \log x \, dx$

Taking $u = \log x$ and $v = x^2$ and integrating by parts, we get

$$I = \log x \int x^2 dx - \int \left\{ \left(\frac{d}{dx} \log x \right) \int x^2 dx \right\} dx$$

$$= \log x . \left(\frac{x^3}{3} \right) - \int \frac{1}{x} . \frac{x^3}{3} dx$$

$$= \frac{x^3 \log x}{3} - \int \frac{x^2}{3} dx$$

$$= \frac{x^3 \log x}{3} - \frac{x^3}{9} + C$$

Question 7:

$x \sin^{-1} x$

Solution:

Let $I = \int x \sin^{-1} x \, dx$

Taking $u = \sin^{-1} x$ and $v = x$ and integrating by parts, we get

$$I = \sin^{-1} x \int x \, dx - \int \left\{ \left(\frac{d}{dx} \sin^{-1} x \right) \int x \, dx \right\} dx$$

$$= \sin^{-1} x \left(\frac{x^2}{2} \right) - \int \frac{1}{\sqrt{1-x^2}} \cdot \frac{x^2}{2} \, dx$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \frac{-x^2}{\sqrt{1-x^2}} \, dx$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \left\{ \frac{1-x^2}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} \right\} dx$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \left\{ \sqrt{1-x^2} - \frac{1}{\sqrt{1-x^2}} \right\} dx$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \left\{ \int \sqrt{1-x^2} \, dx - \int \frac{1}{\sqrt{1-x^2}} \, dx \right\}$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \left\{ \frac{x}{2} \sqrt{1-x^2} + \frac{1}{2} \sin^{-1} x - \sin^{-1} x \right\} + C$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{x}{4} \sqrt{1-x^2} + \frac{1}{4} \sin^{-1} x - \frac{1}{2} \sin^{-1} x + C$$

$$= \frac{1}{4} \left(2x^2 - 1 \right) \sin^{-1} x + \frac{x}{4} \sqrt{1-x^2} + C$$

Question 8:

$x \tan^{-1} x$

Solution:

Let $I = \int x \tan^{-1} x \, dx$

Taking $u = \tan^{-1} x$ and $v = x$ and integrating by parts, we get

$$I = \tan^{-1} x \int x dx - \int \left\{ \left(\frac{d}{dx} \tan^{-1} x \right) \int x dx \right\} dx$$

$$= \tan^{-1} x \left(\frac{x^2}{2} \right) - \int \frac{1}{1+x^2} \cdot \frac{x^2}{2} dx$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} \int \frac{x^2}{1+x^2} dx$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} \int \left\{ \frac{x^2+1}{1+x^2} - \frac{1}{1+x^2} \right\} dx$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2} \right) dx$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} \left(x - \tan^{-1} x \right) + C$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{x}{2} + \frac{1}{2} \tan^{-1} x + C$$

Question 9:

$$x \cos^{-1} x$$

Solution:

Let $I = \int x \cos^{-1} x dx$

Taking $u = \cos^{-1} x$ and $v = x$ and integrating by parts, we get

$$I = \cos^{-1} x \int x dx - \int \left\{ \left(\frac{d}{dx} \cos^{-1} x \right) \int x dx \right\} dx$$

$$= \cos^{-1} x \left(\frac{x^2}{2} \right) - \int \frac{-1}{\sqrt{1-x^2}} \cdot \frac{x^2}{2} dx$$

$$= \frac{x^2 \cos^{-1} x}{2} - \frac{1}{2} \int \frac{1-x^2-1}{\sqrt{1-x^2}} dx$$

$$= \frac{x^2 \cos^{-1} x}{2} - \frac{1}{2} \int \left\{ \sqrt{1-x^2} + \left(\frac{-1}{\sqrt{1-x^2}} \right) \right\} dx$$

$$= \frac{x^2 \cos^{-1} x}{2} - \frac{1}{2} \int \sqrt{1-x^2} dx - \frac{1}{2} \int \left(\frac{-1}{\sqrt{1-x^2}} \right) dx$$

$$= \frac{x^2 \cos^{-1} x}{2} - \frac{1}{2} I_1 - \frac{1}{2} \cos^{-1} x \dots\dots\dots\dots\dots(1)$$

Where, $I_1 = \int \sqrt{1-x^2} \, dx$

$\Rightarrow I_1 = \sqrt{1-x^2} \int 1 \, dx - \int \frac{d}{dx} \sqrt{1-x^2} \int 1 \, dx$

$\Rightarrow I_1 = x\sqrt{1-x^2} - \int \frac{-2x}{2\sqrt{1-x^2}} x \, dx$

$\Rightarrow I_1 = x\sqrt{1-x^2} - \int \frac{-x^2}{\sqrt{1-x^2}} \, dx$

$\Rightarrow I_1 = x\sqrt{1-x^2} - \int \frac{1-x^2-1}{\sqrt{1-x^2}} \, dx$

$\Rightarrow I_1 = x\sqrt{1-x^2} - \left\{ \int \sqrt{1-x^2} \, dx + \int \frac{-dx}{\sqrt{1-x^2}} \right\}$

$\Rightarrow I_1 = x\sqrt{1-x^2} - \left\{ I_1 + \cos^{-1} x \right\}$

$\Rightarrow 2I_1 = x\sqrt{1-x^2} - \cos^{-1} x$

$\therefore I_1 = \frac{x}{2}\sqrt{1-x^2} - \frac{1}{2}\cos^{-1} x$

Substituting in (1),

$I = \frac{x^2 \cos^{-1} x}{2} - \frac{1}{2}\left(\frac{x}{2}\sqrt{1-x^2} - \frac{1}{2}\cos^{-1} x \right) - \frac{1}{2}\cos^{-1} x$

$= \frac{(2x^2 - 1)}{4}\cos^{-1} x - \frac{x}{4}\sqrt{1-x^2} + C$

Question 10:

$\left(\sin^{-1} x \right)^2$

Solution:

Let $I = \int \left(\sin^{-1} x \right)^2 . 1 \, dx$

Taking $u = \left(\sin^{-1} x \right)^2$ and $v = 1$ and integrating by parts, we get

$$I = \int \left(\sin^{-1} x\right)^2 . \int 1 dx - \int \left\{\frac{d}{dx}\left(\sin^{-1} x\right)^2 . \int 1 dx\right\} dx$$

$$= \left(\sin^{-1} x\right)^2 . x - \int \frac{2\sin^{-1} x}{\sqrt{1-x^2}} . x dx$$

$$= x\left(\sin^{-1} x\right)^2 + \int \sin^{-1} x \left(\frac{-2x}{\sqrt{1-x^2}}\right) dx$$

$$= x\left(\sin^{-1} x\right)^2 + \left[\sin^{-1} x \int \frac{-2x}{\sqrt{1-x^2}} dx - \int \left\{\left(\frac{d}{dx}\sin^{-1} x\right)\int \frac{-2x}{\sqrt{1-x^2}} dx\right\} dx\right]$$

$$= x\left(\sin^{-1} x\right)^2 + \left[\sin^{-1} x . 2\sqrt{1-x^2} - \int \frac{1}{\sqrt{1-x^2}} . 2\sqrt{1-x^2} dx\right]$$

$$= x\left(\sin^{-1} x\right)^2 + 2\sqrt{1-x^2} \sin^{-1} x - \int 2 dx$$

$$= x\left(\sin^{-1} x\right)^2 + 2\sqrt{1-x^2} \sin^{-1} x - 2x + C$$

Question 11:

$$\frac{x\cos^{-1} x}{\sqrt{1-x^2}}$$

Solution:

Let $I = \int \dfrac{x\cos^{-1} x}{\sqrt{1-x^2}} dx$

$$I = \frac{-1}{2}\int \frac{-2x}{\sqrt{1-x^2}} . \cos^{-1} x dx$$

Taking $u = \cos^{-1} x$ and $v = \left(\dfrac{-2x}{\sqrt{1-x^2}}\right)$ and integrating by parts, we get

$$I = \frac{-1}{2}\left[\cos^{-1} x \int \frac{-2x}{\sqrt{1-x^2}} dx - \int \left\{\left(\frac{d}{dx}\cos^{-1} x\right)\int \frac{-2x}{\sqrt{1-x^2}} dx\right\} dx\right]$$

$$= \frac{-1}{2}\left[\cos^{-1} x . 2\sqrt{1-x^2} - \int \frac{-1}{\sqrt{1-x^2}} . 2\sqrt{1-x^2} dx\right]$$

$$= \frac{-1}{2}\left[2\sqrt{1-x^2}\cos^{-1} x + \int 2 dx\right]$$

$$= \frac{-1}{2}\left[2\sqrt{1-x^2}\cos^{-1} x + 2x\right] + C$$

$$= -\left[\sqrt{1-x^2}\cos^{-1} x + x\right] + C$$

Question 12:

$$x\sec^2 x$$

Solution:

Let $I = \int x\sec^2 x dx$

Taking $u = x$ and $v = \sec^2 x$ and integrating by parts, we get

$$I = x\int \sec^2 x\,dx - \int\left\{\left(\frac{d}{dx}x\right)\int \sec^2 x\,dx\right\}dx$$

$$= x\tan x - \int 1.\tan x\,dx$$

$$= x\tan x + \log|\cos x| + C$$

Question 13:

$\tan^{-1} x$

Solution:

Let $I = \int 1.\tan^{-1} x\,dx$

Taking $u = \tan^{-1} x$ and $v = 1$ and integrating by parts, we get

$$I = \tan^{-1} x\int 1\,dx - \int\left\{\left(\frac{d}{dx}\tan^{-1}x\right)\int 1.dx\right\}dx$$

$$= \tan^{-1} x.x - \int \frac{1}{1+x^2}x\,dx$$

$$= x\tan^{-1} x - \frac{1}{2}\int \frac{2}{1+x^2}\,dx$$

$$= x\tan^{-1} x - \frac{1}{2}\log\left|1+x^2\right| + C$$

$$= x\tan^{-1} x - \frac{1}{2}\log\left(1+x^2\right) + C$$

Question 14:

$x\left(\log x\right)^2$

Solution:

Let $I = \int x\left(\log x\right)^2 dx$

Taking $u = \left(\log x\right)^2$ and $v = x$ and integrating by parts, we get

$$I = \left(\log x\right)^2\int x\,dx - \int\left[\left\{\frac{d}{dx}\left(\log x\right)^2\right\}\int x\,dx\right]dx$$

$$= \frac{x^2}{2}\left(\log x\right)^2 - \left[\int 2\log x.\frac{1}{x}.\frac{x^2}{2}\,dx\right]$$

$$= \frac{x^2}{2}\left(\log x\right)^2 - \int x\log x\,dx$$

Again, using integration by parts, we get

$$I = \frac{x^2}{2}(\log x)^2 - \left[\log x \int x\,dx - \int \left\{\left(\frac{d}{dx}\log x\right)\int x\,dx\right\}dx\right]$$

$$= \frac{x^2}{2}(\log x)^2 - \left[\frac{x^2}{2}\log x - \int \frac{1}{x}\cdot\frac{x^2}{2}dx\right]$$

$$= \frac{x^2}{2}(\log x)^2 - \frac{x^2}{2}\log x + \frac{1}{2}\int x\,dx$$

$$= \frac{x^2}{2}(\log x)^2 - \frac{x^2}{2}\log x + \frac{x^2}{4} + C$$

Question 15:

$$\left(x^2+1\right)\log x$$

Solution:

Let $I = \int \left(x^2+1\right)\log x\,dx = \int x^2 \log x\,dx + \int \log x\,dx$

Let $I = I_1 + I_2 \ldots\ldots\ldots(1)$

Where, $I_1 = \int x^2 \log x\,dx$ and $I_2 = \int \log x\,dx$

$I_1 = \int x^2 \log x\,dx$

Taking $u = \log x$ and $v = x^2$ and integrating by parts, we get

$$I_1 = \log x \int x^2\,dx - \int\left[\left(\frac{d}{dx}\log x\right)\int x^2\,dx\right]dx$$

$$= \log x.\frac{x^3}{3} - \int \frac{1}{x}.\frac{x^3}{3}dx$$

$$= \frac{x^3}{3}\log x - \frac{1}{3}\left(\int x^2\,dx\right)$$

$$= \frac{x^3}{3}\log x - \frac{x^3}{9} + C_1 \ldots\ldots\ldots(2)$$

$I_2 = \int \log x\,dx$

Taking $u = \log x$ and $v = 1$ and integrating by parts,

$$I_2 = \log x \int 1.dx - \int\left[\left(\frac{d}{dx}\log x\right)\int 1.dx\right]$$

$$= \log x.x - \int \frac{1}{x}.x\,dx$$

$$= x \log x - \int 1.dx$$

$$= x \log x - x + C_2 \ldots\ldots\ldots(3)$$

Using equations (2) and (3) in (1),

$$I = \frac{x^3}{3}\log x - \frac{x^3}{9} + C_1 + x\log x - x + C_2$$

$$= \frac{x^3}{3}\log x - \frac{x^3}{9} + x\log x - x + (C_1 + C_2)$$

$$= \left(\frac{x^3}{3} + x\right)\log x - \frac{x^3}{9} - x + C$$

Question 16:

$e^x(\sin x + \cos x)$

Solution:

Let $I = \int e^x(\sin x + \cos x)\,dx$

Let $f(x) = \sin x$

$f'(x) = \cos x$

$I = \int e^x\{f(x) + f'(x)\}dx$

Since, $\int e^x\{f(x) + f'(x)\}dx = e^x f(x) + C$

$\therefore I = e^x \sin x + C$

Question 17:

$\dfrac{xe^x}{(1+x)^2}$

Solution:

Let, $I = \int \dfrac{xe^x}{(1+x)^2}dx = \int e^x\left\{\dfrac{x}{(1+x)^2}\right\}dx$

$= \int e^x\left\{\dfrac{1+x-1}{(1+x)^2}\right\}dx = \int e^x\left\{\dfrac{1}{1+x} - \dfrac{1}{(1+x)^2}\right\}dx$

Here, $f(x) = \dfrac{1}{1+x}$ $f'(x) = \dfrac{-1}{(1+x)^2}$

$\Rightarrow \int \dfrac{xe^x}{(1+x)^2}dx = \int e^x\{f(x) + f'(x)\}dx$

Since, $\int e^x\{f(x) + f'(x)\}dx = e^x f(x) + C$

$\therefore \int \dfrac{xe^x}{(1+x)^2}dx = e^x \dfrac{1}{1+x} + C$

Question 18:

$$e^x \left(\frac{1+\sin x}{1+\cos x} \right)$$

Solution:

$$e^x \left(\frac{1+\sin x}{1+\cos x} \right) = e^x \left(\frac{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} + 2\sin \frac{x}{2} \cos \frac{x}{2}}{2\cos^2 \frac{x}{2}} \right)$$

$$= \frac{e^x \left(\sin \frac{x}{2} + \cos \frac{x}{2} \right)^2}{2\cos^2 \frac{x}{2}} = \frac{1}{2} e^x \left(\frac{\sin \frac{x}{2} + \cos \frac{x}{2}}{\cos \frac{x}{2}} \right)^2$$

$$= \frac{1}{2} e^x \left[\tan \frac{x}{2} + 1 \right]^2$$

$$= \frac{1}{2} e^x \left[1 + \tan \frac{x}{2} \right]^2$$

$$= \frac{1}{2} e^x \left[1 + \tan^2 \frac{x}{2} + 2\tan \frac{x}{2} \right]$$

$$= \frac{1}{2} e^x \left[\sec^2 \frac{x}{2} + 2\tan \frac{x}{2} \right]$$

$$\frac{e^x (1+\sin x) dx}{(1+\cos x)} = e^x \left[\frac{1}{2} \sec^2 \frac{x}{2} + \tan \frac{x}{2} \right] \dots \dots (1)$$

Let $\tan \frac{x}{2} = f(x)$ so $f'(x) = \frac{1}{2} \sec^2 \frac{x}{2}$

It is known that, $\int e^x \{ f(x) + f'(x) \} dx = e^x f(x) + C$

From equation (1), we get

$$\int \frac{e^x (1+\sin x)}{(1+\cos x)} dx = e^x \tan \frac{x}{2} + C$$

Question 19:

$$e^x \left(\frac{1}{x} - \frac{1}{x^2} \right)$$

Solution:

Let $I = \int e^x \left(\frac{1}{x} - \frac{1}{x^2} \right) dx$

Here, $\frac{1}{x} = f(x)$ $f'(x) = \frac{-1}{x^2}$

It is known that,

$$\int e^x \{ f(x) + f'(x) \} dx$$

$$= e^x f(x) + C$$

$$\therefore I = \frac{e^x}{x} + C$$

Question 20:

$$\frac{(x-3)e^x}{(x-1)^3}$$

Solution:

$$\int e^x \left\{ \frac{x-3}{(x-1)^3} \right\} dx = \int e^x \left\{ \frac{x-1-2}{(x-1)^3} \right\} dx$$

$$= \int e^x \left\{ \frac{1}{(x-1)^2} - \frac{2}{(x-1)^3} \right\} dx$$

Let $\quad f(x) = \dfrac{1}{(x-1)^2} \quad f'(x) = \dfrac{-2}{(x-1)^3}$

It is known that,

$$\int e^x \{ f(x) + f'(x) \} dx = e^x f(x) + C$$

$$\therefore \int e^x \left\{ \frac{(x-3)}{(x-1)^2} \right\} dx = \frac{e^x}{(x-1)^2} + C$$

Question 21:

$$e^{2x} \sin x$$

Solution:

Let $I = e^{2x} \sin x \, dx \dots\dots\dots(1)$

Taking $u = \sin x$ and $v = e^{2x}$ and integrating by parts, we get

$$I = \sin x \int e^{2x} dx - \int \left\{ \left(\frac{d}{dx} \sin x \right) \int e^{2x} dx \right\} dx$$

$$\Rightarrow I = \sin x . \frac{e^{2x}}{2} - \int \cos x . \frac{e^{2x}}{2} dx$$

$$\Rightarrow I = \frac{e^{2x} \sin x}{2} - \frac{1}{2} \int e^{2x} \cos x \, dx$$

Again, using integration by parts, we get

$$I = \frac{e^{2x} \sin x}{2} - \frac{1}{2}\left[\cos x \int e^{2x} dx - \int \left\{ \left(\frac{d}{dx} \cos x \right) \int e^{2x} dx \right\} dx \right]$$

$$\Rightarrow I = \frac{e^{2x} \sin x}{2} - \frac{1}{2}\left[\cos x . \frac{e^{2x}}{2} - \int (-\sin x) \frac{e^{2x}}{2} dx \right]$$

$$\Rightarrow I = \frac{e^{2x} \sin x}{2} - \frac{1}{2}\left[\frac{e^{2x} \cos x}{2} + \frac{1}{2} \int e^{2x} \sin x \, dx \right]$$

$$\Rightarrow I = \frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{4} - \frac{1}{4} I \qquad \left[\text{From} (1) \right]$$

$$\Rightarrow I + \frac{1}{4} I = \frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{4}$$

$$\Rightarrow \frac{5}{4} I = \frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{4}$$

$$\Rightarrow I = \frac{4}{5}\left[\frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{4} \right] + C$$

$$\Rightarrow I = \frac{e^{2x}}{5}\left[2\sin x - \cos x \right] + C$$

Question 22:

$$\sin^{-1}\left(\frac{2x}{1 + x^2} \right)$$

Solution:

Let $x = \tan \theta \qquad dx = \sec^2 \theta \, d\theta$

$$\therefore \sin^{-1}\left(\frac{2x}{1 + x^2} \right) = \sin^{-1}\left(\frac{2\tan \theta}{1 + \tan^2 \theta} \right) = \sin^{-1}\left(\sin 2\theta \right) = 2\theta$$

$$\int \sin^{-1}\left(\frac{2x}{1 + x^2} \right) dx = \int 2\theta . \sec^2 \theta \, d\theta = 2\int \theta . \sec^2 \theta \, d\theta$$

Using integration by parts, we get

$$2\left[\theta.\int \sec^2\theta \, d\theta - \int\left\{\left(\frac{d}{d\theta}\theta\right)\int \sec^2\theta \, d\theta\right\}d\theta\right]$$

$$= 2\left[\theta.\tan\theta - \int\tan\theta \, d\theta\right]$$

$$= 2\left[\theta.\tan\theta + \log|\cos\theta|\right] + C$$

$$= 2\left[x\tan^{-1}x + \log\left|\frac{1}{\sqrt{1+x^2}}\right|\right] + C$$

$$= 2x\tan^{-1}x + 2\log\left(1+x^2\right)^{\frac{-1}{2}} + C$$

$$= 2x\tan^{-1}x + 2\left[\frac{-1}{2}\log\left(1+x^2\right)\right] + C$$

$$= 2x\tan^{-1}x - \log\left(1+x^2\right) + C$$

Question 23:

$\int x^2 e^{x^3} dx$ equals

A. $\frac{1}{3}e^{x^3} + C$

B. $\frac{1}{3}e^{x^2} + C$

C. $\frac{1}{2}e^{x^3} + C$

D. $\frac{1}{2}e^{x^2} + C$

Solution:

Let $I = \int x^2 e^{x^3} dx$

Also, let $x^3 = t$ so, $3x^2 dx = dt$

$$\Rightarrow I = \frac{1}{3}\int e^t dt$$

$$= \frac{1}{3}\left(e^t\right) + C$$

$$= \frac{1}{3}e^{x^3} + C$$

Thus, the correct option is A.

$\int e^x \sec x (1 + \tan x) dx$ equals

    A. $e^x \cos x + C$
    B. $e^x \sec x + C$
    C. $e^x \sin x + C$
    D. $e^x \tan x + C$

Solution:

$\int e^x \sec x (1 + \tan x) dx$

Consider, $I = \int e^x \sec x (1 + \tan x) dx = \int e^x (\sec x + \sec x \tan x) dx$

Let $\sec x = f(x) \quad \sec x \tan x = f'(x)$

It is known that, $\int e^x \{f(x) + f'(x)\} dx = e^x f(x) + C$

$\therefore I = e^x \sec x + C$

Thus, the correct option is B.

EXERCISE 7.7

Integrate the functions in Exercises 1 to 9.

Question 1:

$\sqrt{4-x^2}$

Solution:

Let $I = \int \sqrt{4-x^2}\,dx = \int \sqrt{(2)^2 - (x)^2}\,dx$

Since, $\sqrt{a^2 - x^2}\,dx = \dfrac{x}{2}\sqrt{a^2 - x^2} + \dfrac{a^2}{2}\sin^{-1}\dfrac{x}{a} + C$

$\therefore I = \dfrac{x}{2}\sqrt{4-x^2} + \dfrac{4}{2}\sin^{-1}\dfrac{x}{2} + C$

$= \dfrac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\dfrac{x}{2} + C$

Question 2:

$\sqrt{1-4x^2}$

Solution:

Let, $I = \int \sqrt{1-4x^2}\,dx = \int \sqrt{(1)^2 - (2x)^2}\,dx$

Put, $2x = t \Rightarrow 2dx = dt$

$\therefore I = \dfrac{1}{2}\int \sqrt{(1)^2 - (t)^2}$

Since, $\sqrt{a^2 - x^2}\,dx = \dfrac{x}{2}\sqrt{a^2 - x^2} + \dfrac{a^2}{2}\sin^{-1}\dfrac{x}{a} + C$

$\Rightarrow I = \dfrac{1}{2}\left[\dfrac{t}{2}\sqrt{1-t^2} + \dfrac{1}{2}\sin^{-1}t\right] + C$

$= \dfrac{t}{4}\sqrt{1-t^2} + \dfrac{1}{4}\sin^{-1}t + C$

$= \dfrac{2x}{4}\sqrt{1-4x^2} + \dfrac{1}{4}\sin^{-1}2x + C$

$= \dfrac{x}{2}\sqrt{1-4x^2} + \dfrac{1}{4}\sin^{-1}2x + C$

Question 3:

$$\sqrt{x^2 + 4x + 6}$$

Solution:

Let $I = \int \sqrt{x^2 + 4x + 6}\,dx$

$= \int \sqrt{x^2 + 4x + 4 + 2}\,dx$

$= \int \sqrt{(x^2 + 4x + 4) + 2}\,dx$

$= \int \sqrt{(x+2)^2 + (\sqrt{2})^2}\,dx$

Since, $\sqrt{x^2 + a^2}\,dx = \dfrac{x}{2}\sqrt{x^2 + a^2} + \dfrac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + C$

$I = \dfrac{(x+2)}{2}\sqrt{x^2 + 4x + 6} + \dfrac{2}{2}\log\left|(x+2) + \sqrt{x^2 + 4x + 6}\right| + C$

$= \dfrac{(x+2)}{2}\sqrt{x^2 + 4x + 6} + \log\left|(x+2) + \sqrt{x^2 + 4x + 6}\right| + C$

Question 4:

$$\sqrt{x^2 + 4x + 1}$$

Solution:

Consider,

$I = \int \sqrt{x^2 + 4x + 1}\,dx$

$= \int \sqrt{(x^2 + 4x + 4) - 3}\,dx$

$= \int \sqrt{(x+2)^2 - (\sqrt{3})^2}\,dx$

Since, $\sqrt{x^2 - a^2}\,dx = \dfrac{x}{2}\sqrt{x^2 - a^2} - \dfrac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C$

$\therefore I = \dfrac{(x+2)}{2}\sqrt{x^2 + 4x + 1} - \dfrac{3}{2}\log\left|(x-2) + \sqrt{x^2 + 4x + 1}\right| + C$

Question 5:

$$\sqrt{1-4x-x^2}$$

Solution:

Consider, $I = \int \sqrt{1-4x-x^2}\, dx$

$= \int \sqrt{1-\left(x^2+4x+4-4\right)}\, dx$

$= \int \sqrt{1+4-\left(x+2\right)^2}\, dx$

$= \int \sqrt{\left(\sqrt{5}\right)^2-\left(x+2\right)^2}\, dx$

Since, $\sqrt{a^2-x^2}\, dx = \dfrac{x}{2}\sqrt{a^2-x^2}+\dfrac{a^2}{2}\sin^{-1}\dfrac{x}{a}+C$

$\therefore I = \dfrac{(x+2)}{2}\sqrt{1-4x-x^2}+\dfrac{5}{2}\sin^{-1}\left(\dfrac{x+2}{\sqrt{5}}\right)+C$

Question 6:

$$\sqrt{x^2+4x-5}$$

Solution:

Let $I = \int \sqrt{x^2+4x-5}\, dx$

$= \int \sqrt{\left(x^2+4x+4\right)-9}\, dx = \int \sqrt{\left(x+2\right)^2-\left(3\right)^2}\, dx$

Since, $\int \sqrt{x^2-a^2}\, dx = \dfrac{x}{2}\sqrt{x^2-a^2}-\dfrac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$

$\therefore I = \dfrac{(x+2)}{2}\sqrt{x^2+4x-5}-\dfrac{9}{2}\log\left|(x+2)+\sqrt{x^2+4x-5}\right|+C$

Question 7:

$$\sqrt{1+3x-x^2}$$

Solution:

Put, $I = \int \sqrt{1+3x-x^2}\, dx$

$= \int \sqrt{1-\left(x^2-3x+\dfrac{9}{4}-\dfrac{9}{4}\right)}\, dx$

$= \int \sqrt{\left(1+\dfrac{9}{4}\right)-\left(x-\dfrac{3}{2}\right)^2}\, dx = \int \sqrt{\left(\dfrac{\sqrt{13}}{2}\right)^2-\left(x-\dfrac{3}{2}\right)^2}\, dx$

Since, $\int \sqrt{a^2 - x^2} dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C$

$\therefore I = \frac{x - \frac{3}{2}}{2} \sqrt{1 + 3x - x^2} + \frac{13}{4 \times 2} \sin^{-1} \left(\frac{x - \frac{3}{2}}{\frac{\sqrt{13}}{2}} \right) + C$

$= \frac{2x - 3}{4} \sqrt{1 + 3x - x^2} + \frac{13}{8} \sin^{-1} \left(\frac{2x - 3}{\sqrt{13}} \right) + C$

Question 8:

$\sqrt{x^2 + 3x}$

Solution:

Let $I = \int \sqrt{x^2 + 3x} \, dx$

$= \int \sqrt{x^2 + 3x + \frac{9}{4} - \frac{9}{4}} \, dx$

$= \int \sqrt{\left(x + \frac{3}{2} \right)^2 - \left(\frac{3}{2} \right)^2} \, dx$

Since, $\sqrt{x^2 - a^2} dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log \left| x + \sqrt{x^2 - a^2} \right| + C$

$\therefore I = \frac{\left(x + \frac{3}{2} \right)}{2} \sqrt{x^2 + 3x} - \frac{\frac{9}{4}}{2} \log \left| \left(x + \frac{3}{2} \right) + \sqrt{x^2 + 3x} \right| + C$

$= \frac{(2x + 3)}{4} \sqrt{x^2 + 3x} - \frac{9}{8} \log \left| \left(x + \frac{3}{2} \right) + \sqrt{x^2 + 3x} \right| + C$

Question 9:

$\sqrt{1 + \frac{x^2}{9}}$

Solution:

Let $I = \int \sqrt{1 + \frac{x^2}{9}} \, dx = \frac{1}{3} \int \sqrt{9 + x^2} \, dx = \frac{1}{3} \int \sqrt{(3)^2 + x^2} \, dx$

Since, $\sqrt{x^2 + a^2} dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2} \log \left| x + \sqrt{x^2 + a^2} \right| + C$

$\therefore I = \frac{1}{3} \left[\frac{x}{2}\sqrt{x^2 + 9} + \frac{9}{2} \log \left| x + \sqrt{x^2 + 9} \right| \right] + C$

$= \frac{x}{6}\sqrt{x^2 + 9} + \frac{3}{2} \log \left| x + \sqrt{x^2 + 9} \right| + C$

Question 10:

$\int \sqrt{1+x^2}$ is equal to

A. $\dfrac{x}{2}\sqrt{1+x^2} + \dfrac{1}{2}\log\left|x + \sqrt{1+x^2}\right| + C$

B. $\dfrac{2}{3}\left(1+x^2\right)^{\frac{2}{3}} + C$

C. $\dfrac{2}{3}x\left(1+x^2\right)^{\frac{2}{3}} + C$

D. $\dfrac{x^3}{2}\sqrt{1+x^2} + \dfrac{1}{2}x^2\log\left|x + \sqrt{1+x^2}\right| + C$

Solution:

Since. $\sqrt{a^2+x^2}\,dx = \dfrac{x}{2}\sqrt{a^2+x^2} + \dfrac{a^2}{2}\log\left|x + \sqrt{x^2+a^2}\right| + C$

$\therefore \int \sqrt{1+x^2}\,dx = \dfrac{x}{2}\sqrt{1+x^2} + \dfrac{1}{2}\log\left|x + \sqrt{1+x^2}\right| + C$

Thus, the correct option is A.

Question 11:

$\int \sqrt{x^2 - 8x + 7}\,dx$ is equal to

A. $\dfrac{1}{2}(x-4)\sqrt{x^2-8x+7} + 9\log\left|x-4+\sqrt{x^2-8x+7}\right| + C$

B. $\dfrac{1}{2}(x+4)\sqrt{x^2-8x+7} + 9\log\left|x+4+\sqrt{x^2-8x+7}\right| + C$

C. $\dfrac{1}{2}(x-4)\sqrt{x^2-8x+7} - 3\sqrt{2}\log\left|x-4+\sqrt{x^2-8x+7}\right| + C$

D. $\dfrac{1}{2}(x-4)\sqrt{x^2-8x+7} - \dfrac{9}{2}\log\left|x-4+\sqrt{x^2-8x+7}\right| + C$

Solution:

Let $I = \int \sqrt{x^2-8x+7}\,dx$

$= \int \sqrt{\left(x^2-8x+16\right)-9}\,dx$

$= \int \sqrt{\left(x-4\right)^2-\left(3\right)^2}\,dx$

Since, $\sqrt{x^2-a^2}\,dx = \dfrac{x}{2}\sqrt{x^2-a^2} - \dfrac{a^2}{2}\log\left|x + \sqrt{x^2-a^2}\right| + C$

$\therefore I = \dfrac{(x-4)}{2}\sqrt{x^2-8x+7} - \dfrac{9}{2}\log\left|(x-4) + \int\sqrt{x^2-8x+7}\right| + C$

Thus, the correct option is D.

EXERCISE 7.8

Evaluate the following definite integrals as limit of sums.

Question 1:

$$\int_a^b x\,dx$$

Solution:

Since, $\int_a^b f(x)dx = (b-a)\lim\limits_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+\ldots+f(a+(n-1)h)\right]$ where $h = \frac{b-a}{n}$

Here, $a = a, b = b$ and $f(x) = x$

$\therefore \int_a^b x\,dx = (b-a)\lim\limits_{n\to\infty}\frac{1}{n}\left[a+(a+h)\ldots(a+2h)\ldots a+(n-1)h\right]$

$= (b-a)\lim\limits_{n\to\infty}\frac{1}{n}\left[\left(\underbrace{a+a+a+\ldots+a}_{n\,times}\right)+(h+2h+3h+\ldots(n-1)h)\right]$

$= (b-a)\lim\limits_{n\to\infty}\frac{1}{n}\left[na+h(1+2+3+\ldots+(n-1))\right]$

$= (b-a)\lim\limits_{n\to\infty}\frac{1}{n}\left[na+h\left\{\frac{(n-1)(n)}{2}\right\}\right]$

$= (b-a)\lim\limits_{n\to\infty}\frac{1}{n}\left[na+\frac{n(n-1)h}{2}\right] = (b-a)\lim\limits_{n\to\infty}\frac{n}{n}\left[a+\frac{(n-1)h}{2}\right]$

$= (b-a)\lim\limits_{n\to\infty}\left[a+\frac{(n-1)h}{2}\right] = (b-a)\lim\limits_{n\to\infty}\left[a+\frac{(n-1)(b-a)}{2n}\right]$

$= (b-a)\lim\limits_{n\to\infty}\left[a+\frac{\left(1-\frac{1}{n}\right)(b-a)}{2}\right] = (b-a)\left[a+\frac{(b-a)}{2}\right]$

$= (b-a)\left[\frac{2a+b-a}{2}\right]$

$= \frac{(b-a)(b+a)}{2}$

$= \frac{1}{2}\left(b^2-a^2\right)$

Question 2:

$\int_{0}^{b}(x+1)\,dx$

Solution:

Let $I = \int_{0}^{b}(x+1)\,dx$

Since, $\int_{a}^{b} f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+\ldots+f(a+(n-1)h)\right]$, where $h = \dfrac{b-a}{n}$

Here, $a = 0, b = 5$ and $f(x) = (x+1)$

$\Rightarrow h = \dfrac{5-0}{n} = \dfrac{5}{n}$

$\therefore \int_{0}^{5}(x+1)\,dx = (5-0)\lim_{n\to\infty}\frac{1}{n}\left[f(0)+f\left(\frac{5}{n}\right)+\ldots+f\left((n-1)\frac{5}{n}\right)\right]$

$= 5\lim_{n\to\infty}\frac{1}{n}\left[1+\left(\frac{5}{n}+1\right)+\ldots\left\{1+\left(\frac{5(n-1)}{n}\right)\right\}\right]$

$= 5\lim_{n\to\infty}\frac{1}{n}\left[\left(\underbrace{1+1+1\ldots1}_{n\ times}\right)+\left[\frac{5}{n}+2.\frac{5}{n}+3.\frac{5}{n}+\ldots+(n-1)\frac{5}{n}\right]\right]$

$= 5\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{5}{n}\{1+2+3\ldots(n-1)\}\right]$

$= 5\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{5}{n}.\frac{(n-1)n}{2}\right] = 5\lim_{n\to\infty}\left[1+\frac{5(n-1)}{2n}\right]$

$= 5\lim_{n\to\infty}\left[1+\frac{5}{2}\left(1-\frac{1}{n}\right)\right] = 5\left[1+\frac{5}{2}\right]$

$= 5\left[\frac{7}{2}\right]$

$= \frac{35}{2}$

Question 3:

$\int_{2}^{3}x^{2}\,dx$

Solution:

Since,

$\int_{a}^{b} f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+\ldots+f(a+(n-1)h)\right]$, where $h = \dfrac{b-a}{n}$

Here, $a = 2, b = 3$ and $f(x) = x^{2}$

$$\Rightarrow h = \frac{3-2}{n} = \frac{1}{n}$$

$$\therefore \int_2^3 x^2 dx = (3-2) \lim_{n \to \infty} \frac{1}{n} \left[f(2) + f\left(2 + \frac{1}{n}\right) + f\left(2 + \frac{2}{n}\right) \ldots f\left\{2 + (n-1)\frac{1}{n}\right\} \right]$$

$$= \lim_{n \to \infty} \frac{1}{n} \left[(2)^2 + \left(2 + \frac{1}{n}\right)^2 + \left(2 + \frac{2}{n}\right)^2 + \ldots \left(2 + \frac{(n-1)^2}{n}\right) \right]$$

$$= \lim_{n \to \infty} \frac{1}{n} \left[2^2 + \left\{2^2 + \left(\frac{1}{n}\right)^2 + 2.2.\frac{1}{n}\right\} + \ldots + \left\{(2)^2 + \frac{(n-1)^2}{n^2} + 2.2.\frac{(n-1)}{n}\right\} \right]$$

$$= \lim_{n \to \infty} \frac{1}{n} \left[\underbrace{\left(2^2 + \ldots + 2^2\right)}_{n \, times} + \left\{\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \ldots + \left(\frac{n-1}{n}\right)^2\right\} + 2.2.\left\{\frac{1}{n} + \frac{2}{n} + \frac{3}{n} + \ldots + \frac{(n-1)}{n}\right\} \right]$$

$$= \lim_{n \to \infty} \frac{1}{n} \left[4n + \frac{1}{n^2}\left\{1^2 + 2^2 + 3^2 \ldots + (n-1)^2\right\} + \frac{4}{n}\left\{1 + 2 + \ldots + (n-1)\right\} \right]$$

$$= \lim_{n \to \infty} \frac{1}{n} \left[4n + \frac{1}{n^2}\left\{\frac{n(n-1)(2n-1)}{6}\right\} + \frac{4}{n}\left\{\frac{n(n-1)}{2}\right\} \right]$$

$$= \lim_{n \to \infty} \frac{1}{n} \left[4n + \frac{n\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right)}{6} + \frac{4n-4}{2} \right] = \lim_{n \to \infty} \left[4 + \frac{1}{6}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right) + 2 - \frac{2}{n} \right]$$

$$= 4 + \frac{2}{6} + 2$$

$$= \frac{19}{3}$$

Question 4:

$$\int_1^4 \left(x^2 - x\right) dx$$

Solution:

Let $I = \int_1^4 \left(x^2 - x\right) dx$

$$= \int_1^4 x^2 dx - \int_1^4 x dx$$

Let $I = I_1 - I_2$, where $I_1 = \int_1^4 x^2 dx$ and $I_2 = \int_1^4 x dx$ ...(1)

Since, $\int_a^b f(x) dx = (b-a) \lim_{n \to \infty} \frac{1}{n} \left[f(a) + f(a+h) + f(a+(n-1)h) \right]$, where $h = \frac{b-a}{n}$

For, $I_1 = \int_1^4 x^2 dx$,

$a = 1, b = 4$ and $f(x) = x^2$

$\therefore h = \dfrac{4-1}{n} = \dfrac{3}{n}$

$I_1 = \displaystyle\int_1^4 x^2 dx = (4-1)\lim_{n\to\infty}\frac{1}{n}\left[f(1)+f(1+h)+\ldots+f(1+(n-1)h)\right]$

$= 3\lim_{n\to\infty}\frac{1}{n}\left[1^2+\left(1+\frac{3}{n}\right)^2+\left(1+2.\frac{3}{n}\right)^2+\ldots\left(1+\frac{(n-1)3}{n}\right)^2\right]$

$= 3\lim_{n\to\infty}\frac{1}{n}\left[1^2+\left\{1^2+\left(\frac{3}{n}\right)^2+2.\frac{3}{n}\right\}+\ldots+\left\{1^2+\left(\frac{(n-1)3}{n}\right)^2+\frac{2.(n-1).3}{2}\right\}\right]$

$= 3\lim_{n\to\infty}\frac{1}{n}\left[\underbrace{\left(1^2+\ldots+1^2\right)}_{n\ times}+\left(\frac{3}{n}\right)^2\left\{1^2+2^2+\ldots+(n-1)^2\right\}+2.\frac{3}{n}\left\{1+2+\ldots+(n-1)\right\}\right]$

$= 3\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{9}{n^2}\left\{\frac{(n-1)(n)(2n-1)}{6}\right\}+\frac{6}{n}\left\{\frac{(n-1)(n)}{2}\right\}\right]$

$= 3\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{9n}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+\frac{6n-6}{2}\right]$

$= 3\lim_{n\to\infty}\left[1+\frac{9}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+3-\frac{3}{n}\right]$

$= 3[1+3+3]$

$= 3[7]$

$I_1 = 21 \quad \ldots(2)$

For $I_2 = \displaystyle\int_1^4 x\,dx$

$a = 1, b = 4$ and $f(x) = x$

$\Rightarrow h = \dfrac{4-1}{n} = \dfrac{3}{n}$

$\therefore I_2 = (4-1)\lim\limits_{n \to \infty} \dfrac{1}{n}\left[f(1) + f(1+h) + \ldots + f(a+(n-1)h) \right]$

$= 3\lim\limits_{n \to \infty} \dfrac{1}{n}\left[1 + (1+h) + \ldots + (1+(n-1)h) \right]$

$= 3\lim\limits_{n \to \infty} \dfrac{1}{n}\left[1 + \left(1+\dfrac{3}{n}\right) + \ldots + \left\{1+(n-1)\dfrac{3}{n}\right\} \right]$

$= 3\lim\limits_{n \to \infty} \dfrac{1}{n}\left[\underbrace{\left(1+1+ \ldots +1\right)}_{n\ times} + \dfrac{3}{n}\left(1+2+\ldots+(n-1)\right) \right]$

$= 3\lim\limits_{n \to \infty} \dfrac{1}{n}\left[n + \dfrac{3}{n}\left\{ \dfrac{(n-1)n}{2} \right\} \right]$

$= 3\lim\limits_{n \to \infty}\left[1 + \dfrac{3}{2}\left(1-\dfrac{1}{n}\right) \right]$

$= 3\left[1 + \dfrac{3}{2} \right] = 3\left[\dfrac{5}{2} \right]$

$I_2 = \dfrac{15}{2}$ ...(3)

From equations (2) and (3), we get

$I = I_1 - I_2 = 21 - \dfrac{15}{2} = \dfrac{27}{2}$

Question 5:

$\int_{-1}^{1} e^x dx$

Solution:

Let $I = \int_{-1}^{1} e^x dx$ ...(1)

Since, $\int_{a}^{b} f(x)dx = (b-a)\lim\limits_{n \to \infty}\dfrac{1}{n}\left[f(a) + f(a+h) + f(a+(n-1)h) \right]$, where $h = \dfrac{b-a}{n}$

Here, $a = -1, b = 1$ and $f(x) = e^x$

$$\therefore h = \frac{1+1}{n} = \frac{2}{n}$$

$$\therefore I = (1+1)\lim_{n\to\infty}\frac{1}{n}\left[f(-1)+f\left(-1+\frac{2}{n}\right)+f\left(-1+2.\frac{2}{n}\right)+\ldots+f\left(-1+\frac{(n-1)2}{n}\right)\right]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}\left[e^{-1}+e^{\left(-1+\frac{2}{n}\right)}+e^{\left(-1+2\frac{2}{n}\right)}+e^{\left(-1+(n-1)\frac{2}{n}\right)}\right]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}\left[e^{-1}\left\{1+e^{\frac{2}{n}}+e^{\frac{4}{n}}+e^{\frac{6}{n}}+e^{(n-1)\frac{2}{n}}\right\}\right]$$

$$= 2\lim_{n\to\infty}\frac{e^{-1}}{n}\left[\frac{e^{\frac{2n}{n}}-1}{e^{\frac{2}{n}}-1}\right] = e^{-1}\times 2\lim_{n\to\infty}\frac{1}{n}\left[\frac{e^2-1}{e^{\frac{2}{n}}-1}\right]$$

$$= \frac{e^{-1}\times 2\left(e^2-1\right)}{\lim_{\frac{2}{n}\to 0}\left(\frac{e^{\frac{2}{n}}-1}{\frac{2}{n}}\right)\times 2} \qquad \left[\lim_{h\to 0}\left(\frac{e^h-1}{h}\right)=1\right]$$

$$= \frac{e^2-1}{e}$$

$$= \left(e-\frac{1}{e}\right)$$

Question 6:

$$\int_0^4\left(x+e^{2x}\right)dx$$

Solution:

Since,

$$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+\ldots+f(a+(n-1)h)\right], \text{ where } h=\frac{b-a}{n}$$

Here, $a=0, b=4$ and $f(x)=x+e^{2x}$

$$\therefore h = \frac{4-0}{n} = \frac{4}{n}$$

$$\Rightarrow \int_0^4 \left(x + e^{2x}\right) dx = (4-0) \lim_{n \to \infty} \frac{1}{n} \left[f(0) + f(h) + f(2h) + \ldots + f\left((n-1)h\right) \right]$$

$$= 4 \lim_{n \to \infty} \frac{1}{n} \left[\left(0 + e^0\right) + \left(h + e^{2h}\right) + \left(2h + e^{2.2h}\right) + \ldots + \left\{(n-1)h + e^{2(n-1)h}\right\} \right]$$

$$= 4 \lim_{n \to \infty} \frac{1}{n} \left[1 + \left(h + e^{2h}\right) + \left(2h + e^{4h}\right) + \ldots + \left\{(n-1)h + e^{2(n-1)h}\right\} \right]$$

$$= 4 \lim_{n \to \infty} \frac{1}{n} \left[\left\{h + 2h + 3h + \ldots + (n-1)h\right\} + \left(1 + e^{2h} + e^{4h} + \ldots + e^{2(n-1)h}\right) \right]$$

$$= 4 \lim_{n \to \infty} \frac{1}{n} \left[h\left\{1 + 2 + \ldots (n-1)\right\} + \left(\frac{e^{2hn} - 1}{e^{2h} - 1}\right) \right] = 4 \lim_{n \to \infty} \frac{1}{n} \left[\frac{\left[h(n-1)n\right]}{2} + \left(\frac{e^{2hn} - 1}{e^{2h} - 1}\right) \right]$$

$$= 4 \lim_{x \to \infty} \frac{1}{n} \left[\frac{4}{n} \frac{(n-1)n}{2} + \left(\frac{e^8 - 1}{e^{\frac{8}{n}} - 1}\right) \right] = 4(2) + 4 \lim_{n \to \infty} \frac{\left(e^8 - 1\right)}{\left(\dfrac{e^{\frac{8}{n}} - 1}{\dfrac{8}{n}}\right) 8}$$

$$= 8 + \frac{4\left(e^8 - 1\right)}{8} \qquad\qquad \left(\lim_{x \to 0} \frac{e^x - 1}{x} = 1 \right)$$

$$= 8 + \frac{e^8 - 1}{2} = \frac{15 + e^8}{2}$$

EXERCISE 7.9

Evaluate the definite integrals in Exercises 1 to 20.

Question 1:

$$\int_{-1}^{1}(x+1)dx$$

Solution:

Let $I = \int_{-1}^{1}(x+1)dx$

$$\int(x+1)dx = \frac{x^2}{2}+x = F(x)$$

Using second fundamental theorem of calculus, we get

$I = F(1) - F(-1)$

$= \left(\frac{1}{2}+1\right) - \left(\frac{1}{2}-1\right)$

$= \frac{1}{2}+1-\frac{1}{2}+1$

$= 2$

Question 2:

$$\int_{2}^{3}\frac{1}{x}dx$$

Solution:

Let $I = \int_{2}^{3}\frac{1}{x}dx$

$$\int\frac{1}{x}dx = \log|x| = F(x)$$

Using second fundamental theorem of calculus, we get

$I = F(3) - F(2)$

$= \log|3| - \log|2| = \log\frac{3}{2}$

Question 3:

$$\int_{1}^{2}\left(4x^3 - 5x^2 + 6x + 9\right)dx$$

Solution:

Let $I = \int_{1}^{2}\left(4x^3 - 5x^2 + 6x + 9\right)dx$

$$\int \left(4x^3 - 5x^2 + 6x + 9\right)dx = 4\left(\frac{x^4}{4}\right) - 5\left(\frac{x^3}{3}\right) + 6\left(\frac{x^2}{2}\right) + 9(x)$$

$$= x^4 - \frac{5x^3}{3} + 3x^2 + 9x = F(x)$$

Using second fundamental theorem of calculus, we get

$$I = F(2) - F(1)$$

$$I = \left\{2^4 - \frac{5(2)^3}{3} + 3(2)^2 + 9(2)\right\} - \left\{(1)^4 - \frac{5(1)^3}{3} + 3(1)^2 + 9(1)\right\}$$

$$= \left(16 - \frac{40}{3} + 12 + 18\right) - \left(1 - \frac{5}{3} + 3 + 9\right)$$

$$= 16 - \frac{40}{3} + 12 + 18 - 1 + \frac{5}{3} - 3 - 9$$

$$= 33 - \frac{35}{3}$$

$$= \frac{99 - 35}{3}$$

$$= \frac{64}{3}$$

Question 4:

$$\int_0^{\frac{\pi}{4}} \sin 2x\, dx$$

Solution:

Let $I = \int_0^{\frac{\pi}{4}} \sin 2x\, dx$

$$\int \sin 2x\, dx = \left(\frac{-\cos 2x}{2}\right) = F(x)$$

Using second fundamental theorem of calculus, we get

$$I = F\left(\frac{\pi}{4}\right) - F(0)$$

$$= -\frac{1}{2}\left[\cos 2\left(\frac{\pi}{4}\right) - \cos 0\right] = -\frac{1}{2}\left[\cos\left(\frac{\pi}{2}\right) - \cos 0\right]$$

$$= -\frac{1}{2}[0 - 1]$$

$$= \frac{1}{2}$$

Question 5:

$\int_{0}^{\frac{\pi}{2}} \cos 2x \, dx$

Solution:

Let $I = \int_{0}^{\frac{\pi}{2}} \cos 2x \, dx$

$\int \cos 2x \, dx = \left(\frac{\sin 2x}{2}\right) = F(x)$

Using second fundamental theorem of calculus, we get

$I = F\left(\frac{\pi}{2}\right) - F(0)$

$= \frac{1}{2}\left[\sin 2\left(\frac{\pi}{2}\right) - \sin 0\right] = \frac{1}{2}[\sin \pi - \sin 0]$

$= \frac{1}{2}[0-0] = 0$

Question 6:

$\int_{4}^{5} e^{x} \, dx$

Solution:

Let $I = \int_{4}^{5} e^{x} \, dx$

$\int e^{x} \, dx = e^{x} = F(x)$

Using second fundamental theorem of calculus, we get

$I = F(5) - F(4)$

$= e^{5} - e^{4}$

$= e^{4}(e-1)$

Question 7:

$\int_{0}^{\frac{\pi}{4}} \tan x \, dx$

Solution:

Let $I = \int_{0}^{\frac{\pi}{4}} \tan x \, dx$

$\int \tan x \, dx = -\log|\cos x| = F(x)$

Using second fundamental theorem of calculus, we get

$$I = F\left(\frac{\pi}{4}\right) - F(0)$$

$$= -\log\left|\cos\frac{\pi}{4}\right| + \log\left|\cos 0\right| = -\log\left|\frac{1}{\sqrt{2}}\right| + \log|1|$$

$$= -\log\left(2\right)^{-\frac{1}{2}}$$

$$= \frac{1}{2}\log 2$$

Question 8:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos ec x\, dx$$

Solution:

Let $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos ec x\, dx$

$$\int \cos ec x\, dx = \log\left|\cos ec x - \cot x\right| = F(x)$$

Using second fundamental theorem of calculus, we get

$$I = F\left(\frac{\pi}{4}\right) - F\left(\frac{\pi}{6}\right)$$

$$= \log\left|\cos ec\frac{\pi}{4} - \cot\frac{\pi}{4}\right| - \log\left|\cos ec\frac{\pi}{6} - \cot\frac{\pi}{6}\right|$$

$$\log\left|\sqrt{2} - 1\right| - \log\left|2 - \sqrt{3}\right| = \log\left(\frac{\sqrt{2} - 1}{2 - \sqrt{3}}\right)$$

Question 9:

$$\int_0^1 \frac{dx}{\sqrt{1 - x^2}}$$

Solution:

Let $I = \int_0^1 \frac{dx}{\sqrt{1 - x^2}}$

$$\int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x = F(x)$$

Using second fundamental theorem of calculus, we get

$$I = F(1) - F(0)$$
$$= \sin^{-1}(1) - \sin^{-1}(0)$$
$$= \frac{\pi}{2} - 0$$
$$= \frac{\pi}{2}$$

Question 10:

$$\int_0^1 \frac{dx}{1 + x^2}$$

Solution:

Let $I = \int_0^1 \frac{dx}{1 + x^2}$

$$\int \frac{dx}{1 + x^2} = \tan^{-1} x = F(x)$$

Using second fundamental theorem of calculus, we get

$$I = F(1) - F(0)$$
$$= \tan^{-1}(1) - \tan^{-1}(0)$$
$$= \frac{\pi}{4}$$

Question 11:

$$\int_2^3 \frac{dx}{x^2 - 1}$$

Solution:

Let $I = \int_2^3 \frac{dx}{x^2 - 1}$

$$\int \frac{dx}{x^2 - 1} = \frac{1}{2} \log \left| \frac{x-1}{x+1} \right| = F(x)$$

Using second fundamental theorem of calculus, we get

$$I = F(3) - F(2)$$
$$= \frac{1}{2} \left[\log \left| \frac{3-1}{3+1} \right| - \log \left| \frac{2-1}{2+1} \right| \right] = \frac{1}{2} \left[\log \left| \frac{2}{4} \right| - \log \left| \frac{1}{3} \right| \right]$$
$$= \frac{1}{2} \left[\log \frac{1}{2} - \log \frac{1}{3} \right]$$
$$= \frac{1}{2} \left[\log \frac{3}{2} \right]$$

Question 12:

$\int_0^{\frac{\pi}{2}} \cos^2 x\, dx$

Solution:

Let $I = \int_0^{\frac{\pi}{2}} \cos^2 x\, dx$

$\int \cos^2 x\, dx = \int \left(\frac{1 + \cos 2x}{2} \right) dx = \frac{x}{2} + \frac{\sin 2x}{4} = \frac{1}{2} \left(x + \frac{\sin 2x}{2} \right) = F(x)$

Using second fundamental theorem of calculus, we get

$I = \left[F\left(\frac{\pi}{2} \right) - F(0) \right] = \frac{1}{2} \left[\left(\frac{\pi}{2} + \frac{\sin \pi}{2} \right) - \left(0 + \frac{\sin \pi}{2} \right) \right]$

$= \frac{1}{2} \left[\frac{\pi}{2} + 0 - 0 - 0 \right]$

$= \frac{\pi}{4}$

Question 13:

$\int_2^3 \frac{x}{x^2 + 1} dx$

Solution:

Let $I = \int_2^3 \frac{x}{x^2 + 1} dx$

$\int \frac{x}{x^2 + 1} dx = \frac{1}{2} \int \frac{2x}{x^2 + 1} dx = \frac{1}{2} \log(1 + x^2) = F(x)$

Using second fundamental theorem of calculus, we get

$I = F(3) - F(2)$

$= \frac{1}{2} \left[\log\left(1 + (3)^2\right) - \log\left(1 + (2)^2\right) \right]$

$= \frac{1}{2} \left[\log(10) - \log(5) \right]$

$= \frac{1}{2} \log\left(\frac{10}{5} \right) = \frac{1}{2} \log 2$

Question 14:

$\int_0^1 \frac{2x + 3}{5x^2 + 1} dx$

Solution:

Let $I = \int_0^1 \frac{2x + 3}{5x^2 + 1} dx$

$$\int \frac{2x+3}{5x^2+1}dx = \frac{1}{5}\int \frac{5(2x+3)}{5x^2+1}dx = \frac{1}{5}\int \frac{10x+15}{5x^2+1}dx$$

$$= \frac{1}{5}\int \frac{10x}{5x^2+1}dx + 3\int \frac{1}{5x^2+1}dx$$

$$= \frac{1}{5}\int \frac{10x}{5x^2+1}dx + 3\int \frac{1}{5\left(x^2+\frac{1}{5}\right)}dx = \frac{1}{5}\log(5x^2+1) + \frac{3}{5}\cdot\frac{1}{\frac{1}{\sqrt{5}}}\tan^{-1}\frac{x}{\frac{1}{\sqrt{5}}}$$

$$= \frac{1}{5}\log(5x^2+1) + \frac{3}{\sqrt{5}}\tan^{-1}(\sqrt{5})x$$

$$= F(x)$$

Using second fundamental theorem of calculus, we get

$$I = F(1) - F(0)$$

$$= \left\{\frac{1}{5}\log(5+1) + \frac{3}{\sqrt{5}}\tan^{-1}(\sqrt{5})\right\} - \left\{\frac{1}{5}\log(5\times 0+1) + \frac{3}{\sqrt{5}}\tan^{-1}(0)\right\}$$

$$= \frac{1}{5}\log 6 + \frac{3}{\sqrt{5}}\tan^{-1}\sqrt{5}$$

Question 15:

$$\int_0^1 xe^{x^2}dx$$

Solution:

Let $I = \int_0^1 xe^{x^2}dx$

Put, $x^2 = t \Rightarrow 2xdx = dt$

As $x \to 0, t \to 0$ and as $x \to 1, t \to 1$

$$\therefore I = \frac{1}{2}\int_0^1 e^t dt$$

$$\frac{1}{2}\int e^t dt = \frac{1}{2}e^t = F(t)$$

Using second fundamental theorem of calculus, we get

$$I = F(1) - F(0)$$

$$= \frac{1}{2}e - \frac{1}{2}e^0$$

$$= \frac{1}{2}(e-1)$$

Question 16:

$$\int_1^2 \frac{5x^2}{x^2+4x+3}dx$$

Solution:

Let $I = \int_1^2 \frac{5x^2}{x^2+4x+3}dx$

Dividing $5x^2$ by x^2+4x+3, we get

$$I = \int_1^2 \left\{ 5 - \frac{20x+15}{x^2+4x+3} \right\}dx$$

$$= \int_1^2 5\,dx - \int_1^2 \frac{20x+15}{x^2+4x+3}dx$$

$$= \left[5x\right]_1^2 - \int_1^2 \frac{20x+15}{x^2+4x+3}dx$$

$$I = 5 - I_1, \text{ where } I = \int_1^2 \frac{20x+15}{x^2+4x+3}dx \qquad \ldots(1)$$

Let $20x+15 = A\dfrac{d}{dx}\left(x^2+4x+3\right) + B$

$$= 2Ax + (4A+B)$$

Equating the coefficients of x and constant term, we get

$A = 10$ and $B = -25$

Let $x^2+4x+3 = t$

$\Rightarrow (2x+4)dx = dt$

$$\Rightarrow I_1 = 10\int \frac{dt}{t} - 25\int \frac{dx}{(x+2)^2 - 1^2}$$

$$= 10\log t - 25\left[\frac{1}{2}\log\left(\frac{x+2-1}{x+2+1}\right)\right] = \left[10\log\left(x^2+4x+3\right)\right]_1^2 - 25\left[\frac{1}{2}\log\left(\frac{x+1}{x+3}\right)\right]_1^2$$

$$= \left[10\log 15 - 10\log 8\right] - 25\left[\frac{1}{2}\log\frac{3}{5} - \frac{1}{2}\log\frac{2}{4}\right]$$

$$= \left[10\log(5\times 3) - 10\log(4\times 2)\right] - \frac{25}{2}\left[\log 3 - \log 5 - \log 2 + \log 4\right]$$

$$= \left[10\log 5 + 10\log 3 - 10\log 4 - 10\log 2\right] - \frac{25}{2}\left[\log 3 - \log 5 - \log 2 + \log 4\right]$$

$$= \left[10 + \frac{25}{2}\right]\log 5 + \left[-10 - \frac{25}{2}\right]\log 4 + \left[10 - \frac{25}{2}\right]\log 3 + \left[-10 + \frac{25}{2}\right]\log 2$$

$$= \frac{45}{2}\log 5 - \frac{45}{2}\log 4 - \frac{5}{2}\log 3 + \frac{5}{2}\log 2$$

$$= \frac{45}{2}\log\frac{5}{4} - \frac{5}{2}\log\frac{3}{2}$$

Substituting the value I_1 in (1), we get

$$I = 5 - \left[\frac{45}{2} \log \frac{5}{4} - \frac{5}{2} \log \frac{3}{2} \right]$$

$$= 5 - \frac{5}{2} \left[9 \log \frac{5}{4} - \log \frac{3}{2} \right]$$

Question 17:

$$\int_0^{\frac{\pi}{4}} \left(2 \sec^2 x + x^3 + 2 \right) dx$$

Solution:

Let $I = \int_0^{\frac{\pi}{4}} \left(2 \sec^2 x + x^3 + 2 \right) dx$

$$\int \left(2 \sec^2 x + x^3 + 2 \right) dx = 2 \tan x + \frac{x^4}{4} + 2x = F(x)$$

Using second fundamental theorem of calculus, we get

$$I = F\left(\frac{\pi}{4} \right) - F(0) = \left\{ \left(2 \tan \frac{\pi}{4} + \frac{1}{4} \left(\frac{\pi}{4} \right)^4 + 2 \left(\frac{\pi}{4} \right) \right) - \left(2 \tan 0 + 0 + 0 \right) \right\}$$

$$= 2 \tan \frac{\pi}{4} + \frac{\pi^4}{4^5} + \frac{\pi}{2}$$

$$= 2 + \frac{\pi}{2} + \frac{\pi^4}{1024}$$

Question 18:

$$\int_0^{\pi} \left(\sin^2 \frac{x}{2} - \cos^2 \frac{x}{2} \right) dx$$

Solution:

Let $I = \int_0^{\pi} \left(\sin^2 \frac{x}{2} - \cos^2 \frac{x}{2} \right) dx = -\int_0^{\pi} \left(\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} \right) dx$

$$= -\int_0^{\pi} \cos x \, dx$$

$$\int \cos x \, dx = \sin x = F(x)$$

Using second fundamental theorem of calculus, we get

$$I = F(\pi) - F(0)$$

$$= \sin \pi - \sin 0$$

$$= 0$$

Question 19:

$$\int_0^2 \frac{6x+3}{x^2+4}dx$$

Solution:

Let $I = \int_0^2 \frac{6x+3}{x^2+4}dx$

$$\int \frac{6x+3}{x^2+4}dx = 3\int \frac{2x+1}{x^2+4}dx$$

$$= 3\int \frac{2x}{x^2+4}dx + 3\int \frac{1}{x^2+4}dx$$

$$= 3\log(x^2+4) + \frac{3}{2}\tan^{-1}\frac{x}{2} = F(x)$$

Using second fundamental theorem of calculus, we get

$$I = F(2) - F(0)$$

$$= \left\{ 3\log(2^2+4) + \frac{3}{2}\tan^{-1}\left(\frac{2}{2}\right) \right\} - \left\{ 3\log(0+4) + \frac{3}{2}\tan^{-1}\left(\frac{0}{2}\right) \right\}$$

$$= 3\log 8 + \frac{3}{2}\tan^{-1}1 - 3\log 4 - \frac{3}{2}\tan^{-1}0$$

$$= 3\log 8 + \frac{3}{2}\left(\frac{\pi}{4}\right) - 3\log 4 - 0$$

$$= 3\log\left(\frac{8}{4}\right) + \frac{3\pi}{8}$$

$$= 3\log 2 + \frac{3\pi}{8}$$

Question 20:

$$\int_0^1 \left(xe^x + \sin\frac{\pi x}{4} \right)dx$$

Solution:

Let $I = \int_0^1 \left(xe^x + \sin\frac{\pi x}{4} \right)dx$

$$\int_0^1 \left(xe^x + \sin \frac{\pi x}{4} \right) dx = x\int e^x dx - \int \left\{ \left(\frac{d}{dx} x \right) \int e^x dx \right\} dx + \left\{ \frac{-\cos \frac{\pi x}{4}}{\frac{\pi}{4}} \right\}$$

$$= xe^x - \int e^x dx - \frac{4}{\pi} \cos \frac{\pi x}{4}$$

$$= xe^x - e^x - \frac{4}{\pi} \cos \frac{\pi x}{4}$$

$$= F(x)$$

Using second fundamental theorem of calculus, we get

$$I = F(1) - F(0)$$

$$= \left(1.e^1 - e^1 - \frac{4}{\pi} \cos \frac{\pi}{4} \right) - \left(0.e^0 - e^0 - \frac{4}{\pi} \cos 0 \right)$$

$$= e - e - \frac{4}{\pi} \left(\frac{1}{\sqrt{2}} \right) + 1 + \frac{4}{\pi} = 1 + \frac{4}{\pi} - \frac{2\sqrt{2}}{\pi}$$

Question 21:

$$\int_1^{\sqrt{3}} \frac{dx}{1+x^2}$$

A. $\dfrac{\pi}{3}$

B. $\dfrac{2\pi}{3}$

C. $\dfrac{\pi}{6}$

D. $\dfrac{\pi}{12}$ equals

Solution:

$$\int \frac{dx}{1+x^2} = \tan^{-1} x = F(x)$$

Using second fundamental theorem of calculus, we get

$$\int_1^{\sqrt{3}} \frac{dx}{1+x^2} = F(\sqrt{3}) - F(1)$$

$$= \tan^{-1} \sqrt{3} - \tan^{-1} 1$$

$$= \frac{\pi}{3} - \frac{\pi}{4}$$

$$= \frac{\pi}{12}$$

Thus, the correct option is D.

Question 22:

$$\int_0^{\frac{2}{3}} \frac{dx}{4+9x^2}$$

A. $\dfrac{\pi}{6}$

B. $\dfrac{\pi}{12}$

C. $\dfrac{\pi}{24}$

D. $\dfrac{\pi}{4}$ equals

Solution:

$$\int \frac{dx}{4+9x^2} = \int \frac{dx}{(2)^2 + (3x)^2}$$

Put $3x = t \Rightarrow 3dx = dt$

$$\therefore \int \frac{dx}{(2)^2 + (3x)^2} = \frac{1}{3}\int \frac{dt}{(2)^2 + t^2}$$

$$= \frac{1}{3}\left[\frac{1}{2}\tan^{-1}\frac{t}{2} \right]$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{3x}{2} \right)$$

$$= F(x)$$

Using second fundamental theorem of calculus, we get

$$\int_0^{\frac{2}{3}} \frac{dx}{4+9x^2} = F\left(\frac{2}{3}\right) - F(0)$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{3}{2}.\frac{2}{3} \right) - \frac{1}{6}\tan^{-1}0$$

$$= \frac{1}{6}\tan^{-1}1 - 0$$

$$= \frac{1}{6} \times \frac{\pi}{4}$$

$$= \frac{\pi}{24}$$

Thus, the correct option is C.

EXERCISE 7.10

Evaluate the integrals in Exercises 1 to 8 using substitution.

Question 1:

$\int_0^1 \dfrac{x}{x^2+1}dx$

Solution:

$\int_0^1 \dfrac{x}{x^2+1}dx$

Put, $x^2+1=t \Rightarrow 2xdx=dt$

When, $x=0, t=1$ and when $x=1, t=2$

$\therefore \int_0^1 \dfrac{x}{x^2+1}dx = \dfrac{1}{2}\int_1^2 \dfrac{dt}{t}$

$= \dfrac{1}{2}\Big[\log|t|\Big]_1^2$

$= \dfrac{1}{2}\big[\log 2 - \log 1\big]$

$= \dfrac{1}{2}\log 2$

Question 2:

$\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\,\cos^5\phi d\phi$

Solution:

Consider, $I = \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\,\cos^5\phi d\phi = \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\,\cos^4\phi\cos\phi d\phi$

Let $\sin\phi = t \Rightarrow \cos\phi d\phi = dt$

When $\phi = 0, t = 0$ and when $\phi = \dfrac{\pi}{2}, t = 1$

$$\therefore I = \int_0^1 \sqrt{t} \left(1 - t^2\right)^2 dt$$

$$= \int_0^1 t^{\frac{1}{2}} \left(1 + t^4 - 2t^2\right) dt$$

$$= \int_0^1 \left[t^{\frac{1}{2}} + t^{\frac{9}{2}} - 2t^{\frac{5}{2}}\right] dt$$

$$= \left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + \frac{t^{\frac{11}{2}}}{\frac{11}{2}} - \frac{2t^{\frac{7}{2}}}{\frac{7}{2}}\right]_0^1$$

$$= \frac{2}{3} + \frac{2}{11} - \frac{4}{7}$$

$$= \frac{154 + 42 - 132}{231} = \frac{64}{231}$$

Question 3:

$$\int_0^1 \sin^{-1}\left(\frac{2x}{1 + x^2}\right) dx$$

Solution:

Consider, $I = \int_0^1 \sin^{-1}\left(\frac{2x}{1 + x^2}\right) dx$

Let $x = \tan\theta \Rightarrow dx = \sec^2\theta \, d\theta$

When $x = 0, \theta = 0$ and when $x = 1, \theta = \dfrac{\pi}{4}$

$$I = \int_0^{\frac{\pi}{4}} \sin^{-1}\left(\frac{2\tan\theta}{1 + \tan^2\theta}\right) \sec^2\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sin^{-1}(\sin 2\theta) \sec^2\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} 2\theta \sec^2\theta \, d\theta$$

$$= 2\int_0^{\frac{\pi}{4}} \theta \sec^2\theta \, d\theta$$

Taking $u = \theta$ and $v = \sec^2\theta$ and integrating by parts, we get

$$I = 2\left[\theta \int \sec^2\theta \, d\theta - \int \left\{\left(\frac{d}{d\theta}\theta\right)\int \sec^2\theta \, d\theta\right\}d\theta\right]_0^{\frac{\pi}{4}}$$

$$= 2\left[\theta \tan\theta - \int \tan\theta \, d\theta\right]_0^{\frac{\pi}{4}}$$

$$= 2\left[\theta \tan\theta + \log|\cos\theta|\right]_0^{\frac{\pi}{4}}$$

$$= 2\left[\frac{\pi}{4}\tan\frac{\pi}{4} + \log\left|\cos\frac{\pi}{4}\right| - \log|\cos 0|\right] = 2\left[\frac{\pi}{4} + \log\left(\frac{1}{\sqrt{2}}\right) - \log 1\right]$$

$$= 2\left[\frac{\pi}{4} - \frac{1}{2}\log 2\right]$$

$$= \frac{\pi}{2} - \log 2$$

Question 4:

$$\int_0^2 x\sqrt{x+2} \quad \left(\text{Put } x+2 = t^2\right)$$

Solution:

$$\int_0^2 x\sqrt{x+2}\,dx$$

Put, $x+2 = t^2 \Rightarrow dx = 2t\,dt$

When $x = 0, t = \sqrt{2}$ and when $x = 2, t = 2$

$$\therefore \int_0^2 x\sqrt{x+2}\,dx = \int_{\sqrt{2}}^2 \left(t^2 - 2\right)\sqrt{t^2}\,2t\,dt$$

$$= 2\int_{\sqrt{2}}^2 \left(t^2 - 2\right)t^2\,dt$$

$$= 2\int_{\sqrt{2}}^2 \left(t^4 - 2t^2\right)dt$$

$$= 2\left[\frac{t^5}{5} - \frac{2t^3}{3}\right]_{\sqrt{2}}^2$$

$$= 2\left[\frac{32}{5} - \frac{16}{3} - \frac{4\sqrt{2}}{5} + \frac{4\sqrt{2}}{3}\right] = 2\left[\frac{96 - 80 - 12\sqrt{2} + 20\sqrt{2}}{15}\right] = 2\left[\frac{16 + 8\sqrt{2}}{15}\right]$$

$$= \frac{16\left(2 + \sqrt{2}\right)}{15}$$

$$= \frac{16\sqrt{2}\left(\sqrt{2} + 1\right)}{15}$$

Question 5:

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} dx$$

Solution:

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} dx$$

Put, $\cos x = t \Rightarrow -\sin x dx = dt$

When $x = 0, t = 1$ and when $x = \frac{\pi}{2}, t = 0$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} dx = -\int_1^0 \frac{dt}{1+t^2}$$

$$= -\left[\tan^{-1} t \right]_1^0$$

$$= -\left[\tan^{-1} 0 - \tan^{-1} 1 \right]$$

$$= -\left[-\frac{\pi}{4} \right]$$

$$= \frac{\pi}{4}$$

Question 6:

$$\int_0^2 \frac{dx}{x+4-x^2}$$

Solution:

$$\int_0^2 \frac{dx}{x+4-x^2} = \int_0^2 \frac{dx}{-\left(x^2 - x - 4\right)}$$

$$= \int \frac{dx}{-\left(x^2 - x + \frac{1}{4} - \frac{1}{4} - 4\right)} = \int_0^2 \frac{dx}{-\left[\left(x - \frac{1}{2}\right)^2 - \frac{17}{4}\right]}$$

$$= \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2}$$

Let $x - \frac{1}{2} = t \Rightarrow dx = dt$

when $x = 0, t = -\frac{1}{2}$ and when $x = 2, t = \frac{3}{2}$

$$\therefore \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2} = \int_{-\frac{1}{2}}^{\frac{3}{2}} \frac{dt}{\left(\frac{\sqrt{17}}{2}\right)^2 - t^2}$$

$$= \left[\frac{1}{2\left(\frac{\sqrt{17}}{2}\right)} \log \frac{\frac{\sqrt{17}}{2}+t}{\frac{\sqrt{17}}{2}-t} \right]_{-\frac{1}{2}}^{\frac{3}{2}} = \frac{1}{\sqrt{17}} \left[\log \frac{\frac{\sqrt{17}}{2}+\frac{3}{2}}{\frac{\sqrt{17}}{2}-\frac{3}{2}} - \frac{\log \frac{\sqrt{17}}{2}-\frac{1}{2}}{\log \frac{\sqrt{17}}{2}+\frac{1}{2}} \right]$$

$$= \frac{1}{\sqrt{17}} \left[\log \frac{\sqrt{17}+3}{\sqrt{17}-3} - \log \frac{\sqrt{17}-1}{\sqrt{17}+1} \right] = \frac{1}{\sqrt{17}} \log \frac{\sqrt{17}+3}{\sqrt{17}-3} \times \frac{\sqrt{17}+1}{\sqrt{17}-1}$$

$$= \frac{1}{\sqrt{17}} \log \left[\frac{17+3+4\sqrt{17}}{17+3-4\sqrt{17}} \right] = \frac{1}{\sqrt{17}} \log \left[\frac{20+4\sqrt{17}}{20-4\sqrt{17}} \right]$$

$$= \frac{1}{\sqrt{17}} \log \left(\frac{5+\sqrt{17}}{5-\sqrt{17}} \right) = \frac{1}{\sqrt{17}} \log \left[\frac{\left(5+\sqrt{17}\right)\left(5+\sqrt{17}\right)}{25-17} \right]$$

$$= \frac{1}{\sqrt{17}} \log \left[\frac{25+17+10\sqrt{17}}{8} \right] = \frac{1}{\sqrt{17}} \log \left(\frac{42+10\sqrt{17}}{8} \right)$$

$$= \frac{1}{\sqrt{17}} \log \left(\frac{21+5\sqrt{17}}{4} \right)$$

Question 7:

$$\int_{-1}^{1} \frac{dx}{x^2+2x+5}$$

Solution:

$$\int_{-1}^{1} \frac{dx}{x^2+2x+5} = \int_{-1}^{1} \frac{dx}{(x^2+2x+1)+4} = \int_{-1}^{1} \frac{dx}{(x+1)^2+(2)^2}$$

Put, $x+1=t \Rightarrow dx=dt$

When $x=-1, t=0$ and when $x=1, t=2$

$$\int_{-1}^{1} \frac{dx}{(x-1)^2+(2)^2} = \int_{0}^{2} \frac{dt}{t^2+2^2}$$

$$= \left[\frac{1}{2} \tan^{-1} \frac{t}{2} \right]_0^2 = \frac{1}{2} \tan^{-1} 1 - \frac{1}{2} \tan^{-1} 0$$

$$= \frac{1}{2} \left(\frac{\pi}{4} \right) = \frac{\pi}{8}$$

Question 8:

$$\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2} \right) e^{2x} dx$$

Solution:

$$\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2} \right) e^{2x} dx$$

Put, $2x = t \Rightarrow 2dx = dt$

When $x = 1, t = 2$ and when $x = 2, t = 4$

$$\therefore \int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2} \right) e^{2x} dx = \frac{1}{2} \int_2^4 \left(\frac{2}{t} - \frac{2}{t^2} \right) e^t dt$$

$$= \int_2^4 \left(\frac{1}{t} - \frac{1}{t^2} \right) e^t dt$$

Let $\frac{1}{t} = f(t)$

Then, $f'(t) = -\frac{1}{t^2}$

$$\Rightarrow \int_2^4 \left(\frac{1}{t} - \frac{1}{t^2} \right) e^t dt = \int_2^4 e^t \left[f(t) + f'(t) \right] dt$$

$$= \left[e^t f(t) \right]_2^4$$

$$= \left[e^t \cdot \frac{1}{t} \right]_2^4$$

$$= \left[\frac{e^t}{t} \right]_2^4$$

$$= \frac{e^4}{4} - \frac{e^2}{2}$$

$$= \frac{e^2 \left(e^2 - 2 \right)}{4}$$

Question 9:

The value of the integral $\int_{\frac{1}{3}}^1 \frac{\left(x - x^3 \right)^{\frac{1}{3}}}{x^4} dx$ is

A. 6

B. 0

C. 3

D. 4

Solution:

Consider, $I = \int_{\frac{1}{3}}^{1} \dfrac{\left(x - x^3\right)^{\frac{1}{3}}}{x^4} dx$

Let $x = \sin\theta \Rightarrow dx = \cos\theta \, d\theta$

When $x = \dfrac{1}{3}, \theta = \sin^{-1}\left(\dfrac{1}{3}\right)$ and when $x = 1, \theta = \dfrac{\pi}{2}$

$\Rightarrow I = \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \dfrac{\left(\sin\theta - \sin^3\theta\right)^{\frac{1}{3}}}{\sin^4\theta} \cos\theta \, d\theta$

$= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \dfrac{(\sin\theta)^{\frac{1}{3}}\left(1 - \sin^2\theta\right)^{\frac{1}{3}}}{\sin^4\theta} \cos\theta \, d\theta = \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \dfrac{(\sin\theta)^{\frac{1}{3}}(\cos\theta)^{\frac{2}{3}}}{\sin^4\theta} \cos\theta \, d\theta$

$= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \dfrac{(\sin\theta)^{\frac{1}{3}}(\cos\theta)^{\frac{2}{3}}}{\sin^2\theta \sin^2\theta} \cos\theta \, d\theta = \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \dfrac{(\cos\theta)^{\frac{5}{3}}}{(\sin\theta)^{\frac{5}{3}}} \csc^2\theta \, d\theta$

$= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} (\cot\theta)^{\frac{5}{3}} \csc^2\theta \, d\theta$

Put $\cot\theta = t \Rightarrow -\csc^2\theta \, d\theta = dt$

When $\theta = \sin^{-1}\left(\dfrac{1}{3}\right), t = 2\sqrt{2}$ and when $\theta = \dfrac{\pi}{2}, t = 0$

$\therefore I = -\int_{2\sqrt{2}}^{0} (t)^{\frac{5}{3}} dt$

$= -\left[\dfrac{3}{8}(t)^{\frac{8}{3}}\right]_{2\sqrt{2}}^{0}$

$= -\dfrac{3}{8}\left[-\left(2\sqrt{2}\right)^{\frac{8}{3}}\right]_{2\sqrt{2}}^{0} = \dfrac{3}{8}\left[\left(\sqrt{8}\right)^{\frac{8}{3}}\right]$

$= \dfrac{3}{8}\left[(8)^{\frac{4}{3}}\right]$

$= \dfrac{3}{8}[16]$

$= 6$

Thus, the correct option is A.

Question 10:

If $f(x) = \int_0^x t \sin t \, dt$, then $f'(x)$ is

A. $\cos x + x \sin x$

B. $x \sin x$

C. $x \cos x$

D. $\sin x + x \cos x$

Solution:

$$f(x) = \int_0^x t \sin t\, dt$$

Using integration by parts, we get

$$f(x) = t \int_0^x \sin t\, dt - \int_0^x \left\{ \left(\frac{d}{dt} t \right) \int \sin t\, dt \right\} dt$$

$$= \left[t(-\cos t) \right]_0^x - \int_0^x (-\cos t)\, dt$$

$$= \left[-t \cos t + \sin t \right]_0^x$$

$$= -x \cos x + \sin x$$

$$\Rightarrow f'(x) = -\left[\{x(-\sin x)\} + \cos x \right] + \cos x$$

$$= x \sin x - \cos x + \cos x$$

$$= x \sin x$$

Thus, the correct option is B.

EXERCISE 7.11

By using the properties of definite integrals, evaluate the integrals in Exercises 1 to 19.

Question 1:

$$\int_0^{\frac{\pi}{2}} \cos^2 x \, dx$$

Solution:

$$I = \int_0^{\frac{\pi}{2}} \cos^2 x \, dx \quad \ldots(1)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \cos^2\left(\frac{\pi}{2} - x\right) dx \qquad \left(\int_0^a f(x) dx = \int_0^a f(a-x) dx\right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \sin^2 x \, dx \quad \ldots(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\frac{\pi}{2}} \left(\sin^2 x + \cos^2 x\right) dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1 . dx$$

$$\Rightarrow 2I = \left[x\right]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

Question 2:

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

Solution:

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

Consider, $\quad I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \quad \ldots(1)$

$$\Rightarrow I = I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}} dx \qquad \left(\int_0^a f(x) dx = \int_0^a f(a-x) dx\right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx \qquad ...(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1.dx$$

$$\Rightarrow 2I = \left[x\right]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

Question 3:

$$\int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x \, dx}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x} dx$$

Solution:

Let
$$I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x \, dx}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x} dx \qquad ...(1)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}}\left(\frac{\pi}{2} - x\right) dx}{\sin^{\frac{3}{2}}\left(\frac{\pi}{2} - x\right) + \cos^{\frac{3}{2}}\left(\frac{\pi}{2} - x\right)} dx \qquad \left(\int_0^a f(x) = \int_0^a f(a-x) dx\right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos^{\frac{3}{2}} x \, dx}{\cos^{\frac{3}{2}} x + \sin^{\frac{3}{2}} x} dx \qquad ...(3)$$

Adding (1) and (2), we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x} dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1.dx \Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

Question 4:

$$\int_0^{\frac{\pi}{2}} \frac{\cos^5 x dx}{\sin^5 x + \cos^5 x} dx$$

Solution:

Consider,

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos^5 x dx}{\sin^5 x + \cos^5 x} dx \quad \dots(1)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos^5\left(\frac{\pi}{2} - x\right) dx}{\sin^5\left(\frac{\pi}{2} - x\right) + \cos^5\left(\frac{\pi}{2} - x\right)} dx \qquad \left(\int_0^a f(x) = \int_0^a f(a-x) dx\right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin^5 x}{\cos^5 x + \sin^5 x} dx \quad \dots(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^5 x + \cos^5 x}{\sin^5 x + \cos^5 x} dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1.dx \Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

Question 5:

$$\int_{-5}^{5} |x+2| dx$$

Solution:

Let $I = \int_{-5}^{5} |x+2| dx$

As, $(x+2) \leq 0$ on $[-5, -2]$ and $(x+2) \geq 0$ on $[-2, 5]$

$$\therefore \int_{-5}^{5} |x+2| \, dx = \int_{-5}^{-2} -(x+2) \, dx + \int_{-2}^{5} (x+2) \, dx \qquad \left(\int_{a}^{b} f(x) = \int_{a}^{c} f(x) + \int_{c}^{b} f(x) \right)$$

$$I = -\left[\frac{x^2}{2} + 2x \right]_{-5}^{-2} + \left[\frac{x^2}{2} + 2x \right]_{-2}^{5}$$

$$= -\left[\frac{(-2)^2}{2} + 2(-2) - \frac{(-5)^2}{2} - 2(-5) \right] + \left[\frac{(5)^2}{2} + 2(5) - \frac{(-2)^2}{2} - 2(-2) \right]$$

$$= -\left[2 - 4 - \frac{25}{2} + 10 \right] + \left[\frac{25}{2} + 10 - 2 + 4 \right]$$

$$= -2 + 4 + \frac{25}{2} - 10 + \frac{25}{2} + 10 - 2 + 4$$

$$= 29$$

Question 6:

$$\int_{2}^{8} |x-5| \, dx$$

Solution:

Consider, $I = \int_{2}^{8} |x-5| \, dx$

As $(x-5) \leq 0$ on $[2,5]$ and $(x-5) \geq 0$ on $[5,8]$

$$I = \int_{2}^{5} -(x-5) \, dx + \int_{5}^{8} (x-5) \, dx \qquad \left(\int_{a}^{b} f(x) = \int_{a}^{c} f(x) + \int_{c}^{b} f(x) \right)$$

$$= -\left[\frac{x^2}{2} - 5x \right]_{2}^{5} + \left[\frac{x^2}{2} - 5x \right]_{5}^{8}$$

$$= -\left[\frac{25}{2} - 25 - 2 + 10 \right] + \left[32 - 40 - \frac{25}{2} + 25 \right] = 9$$

Question 7:

$$\int_{0}^{1} x(1-x)^n \, dx$$

Solution:

Consider, $I = \int_{0}^{1} x(1-x)^n \, dx$

$$\therefore I = \int_0^1 (1-x)(1-(1-x))^n \, dx$$

$$= \int_0^1 (1-x)(x)^n \, dx = \int_0^1 (x^n - x^{n+1}) \, dx$$

$$= \left[\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2} \right]_0^1 \quad \left(\int_1^a f(x) \, dx = \int_0^a f(a-x) \, dx \right)$$

$$= = \left[\frac{1}{n+1} - \frac{1}{n+2} \right]$$

$$= \frac{(n+2)-(n+1)}{(n+1)(n+2)}$$

$$= \frac{1}{(n+1)(n+2)}$$

Question 8:

$$\int_0^{\frac{\pi}{4}} \log(1+\tan x) \, dx$$

Solution:

Let $I = \int_0^{\frac{\pi}{4}} \log(1+\tan x) \, dx$ ...(1)

$$\therefore I = I = \int_0^{\frac{\pi}{4}} \log\left(1+\tan\left(\frac{\pi}{4}-x\right)\right) dx \quad \left(\int_0^a f(x) \, dx = \int_0^a f(a-x) \, dx \right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log\left\{ 1 + \frac{\tan\frac{\pi}{4} - \tan x}{1 + \tan\frac{\pi}{4} \tan x} \right\} dx \quad \left(\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b} \right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log\left\{ 1 + \frac{1-\tan x}{1+\tan x} \right\} dx \Rightarrow I = \int_0^{\frac{\pi}{4}} \log \frac{2}{(1+\tan x)} dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log 2 \, dx - \int_0^{\frac{\pi}{4}} \log(1+\tan x) \, dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log 2 \, dx - I \quad \left[\text{from}(1) \right]$$

$$\Rightarrow 2I = \log 2 [x]_0^{\frac{\pi}{4}}$$

$$\Rightarrow 2I = \log 2 \left[\frac{\pi}{4} - 0 \right]$$

$$I = \frac{\pi}{8} \log 2$$

Question 9:

$$\int_0^2 x\sqrt{2-x}\,dx$$

Solution:

Consider, $I = \int_0^2 x\sqrt{2-x}\,dx$

$I = \int_0^2 (2-x)\sqrt{2-(2-x)}\,dx \quad \left(\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right)$

$= \int_0^2 (2-x)\sqrt{x}\,dx$

$= \int_0^2 \left\{ 2x^{\frac{1}{2}} - x^{\frac{3}{2}} \right\} dx = \left[2\left(\dfrac{x^{\frac{3}{2}}}{\frac{3}{2}}\right) - \dfrac{x^{\frac{5}{2}}}{\frac{5}{2}} \right]_0^2$

$= \left[\dfrac{4}{3}x^{\frac{3}{2}} - \dfrac{2}{5}x^{\frac{5}{2}} \right]_0^2 = \dfrac{4}{3}(2)^{\frac{3}{2}} - \dfrac{2}{5}(2)^{\frac{5}{2}}$

$= \dfrac{4 \times 2\sqrt{2}}{3} - \dfrac{2}{5} \times 4\sqrt{2} = \dfrac{8\sqrt{2}}{3} - \dfrac{8\sqrt{2}}{5}$

$= \dfrac{40\sqrt{2} - 24\sqrt{2}}{15} = \dfrac{16\sqrt{2}}{15}$

Question 10:

$$\int_0^{\frac{\pi}{2}} \left(2\log\sin x - \log\sin 2x\right) dx$$

Solution:

Consider, $I = \int_0^{\frac{\pi}{2}} \left(2\log\sin x - \log\sin 2x\right) dx$

$I = \int_0^{\frac{\pi}{2}} \left(2\log\sin x - \log\left(2\sin x\cos x\right)\right) dx$

$I = \int_0^{\frac{\pi}{2}} \left(2\log\sin x - \log\sin x - \log\cos x - \log 2\right) dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \left\{\log\sin x - \log\cos x - \log 2\right\} \quad \dots(1)$

Since, $\left(\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right)$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \left\{\log\cos x - \log\sin x - \log 2\right\} dx \quad \dots(2)$

Adding (1) and (2), we get

$$2I = \int_0^{\frac{\pi}{2}} \left(-\log 2 - \log 2 \right) dx$$

$$\Rightarrow 2I = -2\log 2 \int_0^{\frac{\pi}{2}} 1 \cdot dx$$

$$\Rightarrow I = -\log 2 \left[\frac{\pi}{2} \right]$$

$$\Rightarrow I = \frac{\pi}{2} \left(-\log 2 \right)$$

$$\Rightarrow I = \frac{\pi}{2} \left[\log \frac{1}{2} \right]$$

$$\Rightarrow I = \frac{\pi}{2} \log \frac{1}{2}$$

Question 11:

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \, dx$$

Solution:

Let $I = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \, dx$

As $\sin^2(-x) = \left(\sin(-x) \right)^2 = \left(-\sin x \right)^2 = \sin^2 x$, therefore $\sin^2 x$ is an even function.

If $f(x)$ is an even function, then $\int_{-a}^{a} f(x) dx = 2\int_0^a f(x) dx$

$$I = 2\int_0^{\frac{\pi}{2}} \sin^2 x \, dx = 2\int_0^{\frac{\pi}{2}} \frac{1 - \cos 2x}{2} dx$$

$$= \int_0^{\frac{\pi}{2}} \left(1 - \cos 2x \right) dx = \left[x - \frac{\sin 2x}{2} \right]_0^{\frac{\pi}{2}}$$

$$= \left[\frac{\pi}{2} - \frac{\sin 2\left(\frac{\pi}{2} \right)}{2} \right] - \left[0 - \frac{\sin 2(0)}{2} \right]$$

$$= \frac{\pi}{2} - \frac{\sin \pi}{2} - 0$$

$$= \frac{\pi}{2}$$

Question 12:

$$\int_0^\pi \frac{xdx}{1+\sin x}$$

Solution:

Let $I = \int_0^\pi \frac{xdx}{1+\sin x}$ ...(1)

$\Rightarrow I = \int_0^\pi \frac{(\pi - x)}{1+\sin(\pi - x)}dx \quad \left(\int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$

$\Rightarrow I = \int_0^\pi \frac{(\pi - x)}{1+\sin x}dx$ ...(2)

Adding (1) and (2), we get

$2I = \int_0^\pi \frac{x}{1+\sin x}dx + \int_0^\pi \frac{\pi - x}{1+\sin x}dx$

$\Rightarrow 2I = \int_0^\pi \frac{\pi}{1+\sin x}dx$

Multiplying and Dividing by $(1 - \sin x)$

$\Rightarrow 2I = \pi \int_0^\pi \frac{(1 - \sin x)}{(1+\sin x)(1-\sin x)}dx$

$\Rightarrow 2I = \pi \int_0^\pi \frac{1 - \sin x}{\cos^2 x}dx$

$\Rightarrow 2I = \pi \int_0^\pi \{\sec^2 x - \tan x \sec x\}dx$

$\Rightarrow 2I = \pi \left[[\tan x]_0^\pi - [\sec x]_0^\pi\right]$

$\Rightarrow 2I = \pi\left[(\tan(\pi) - \tan(0)) - (\sec(\pi) - \sec(0))\right]$

$\Rightarrow 2I = \pi[2]$

$\Rightarrow I = \pi$

Question 13:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 xdx$$

Solution:

Let $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 xdx$ ...(1)

As $\sin^7(-x) = (\sin(-x))^7 = (-\sin x)^7 = -\sin^7 x$, thus $\sin^7 x$ is an odd function.

$f(x)$ is an odd function, then $\int_{-a}^a f(x)dx = 0$

$$\therefore I = \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x \, dx = 0$$

Question 14:

$$\int_0^{2\pi} \cos^5 x \, dx$$

Solution:

Let $I = \int_0^{2\pi} \cos^5 x \, dx$ ...(1)

$\cos^5 (2\pi - x) = \cos^5 x$

We know that,

$\int_0^{2a} f(x) dx = 2 \int_0^a f(x) dx$, if $f(2a-x) = f(x)$

$= 0$ if $f(2a-x) = -f(x)$

$\therefore I = 2 \int_0^{2\pi} \cos^5 x \, dx$

$\Rightarrow I = 2(0) = 0 \qquad \left[\cos^5 (\pi - x) = -\cos^5 x \right]$

Question 15:

$$\int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x}$$

Solution:

Consider, $I = \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx$ ...(1)

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin \left(\frac{\pi}{2} - x \right) - \cos \left(\frac{\pi}{2} - x \right)}{1 + \sin \left(\frac{\pi}{2} - x \right) \cos \left(\frac{\pi}{2} - x \right)} dx \quad \left(\int_0^a f(x) dx = \int_0^a f(a-x) dx \right)$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1 + \sin x \cos x} dx$ ...(2)

Adding (1) and (2), we get

$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \frac{0}{1 + \sin x \cos x} dx \Rightarrow I = 0$

Question 16:

$\int_0^\pi \log(1 + \cos x)dx$

Solution:

Consider, $I = \int_0^\pi \log(1 + \cos x)dx$ ...(1)

$\Rightarrow I = \int_0^\pi \log(1 + \cos(\pi - x))dx$ $\left(\int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$

$\Rightarrow I = \int_0^\pi \log(1 - \cos x)dx$ ...(2)

Adding (1) and (2), we get

$2I = \int_0^\pi \{\log(1 + \cos x) + \log(1 - \cos x)\}dx$

$\Rightarrow 2I = \int_0^\pi \log(1 - \cos^2 x)dx$

$\Rightarrow 2I = \int_0^\pi \log(\sin^2 x)dx$

$\Rightarrow 2I = 2\int_0^\pi \log(\sin x)dx$

$\Rightarrow I = \int_0^\pi \log(\sin x)dx$ ...(3)

$\therefore \sin(\pi - x) = \sin x$

We know that,

$\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$ if $f(2a - x) = f(x)$

$\therefore I = \int_0^{\frac{\pi}{2}} \log \sin x\, dx$ ...(4)

$\Rightarrow I = 2\int_0^{\frac{\pi}{2}} \log \sin\left(\frac{\pi}{2} - x\right)dx = 2\int_0^{\frac{\pi}{2}} \log \cos x\, dx$ ...(5)

Adding (4) and (5), we get

$$2I = \int_0^{\frac{\pi}{2}} \left(\log \sin x + \log \cos x \, dx \right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \left(\log \sin x + \log \cos x + \log 2 - \log 2 \right) dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \left(\log 2 \sin x \cos x - \log 2 \right) dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \log \sin 2x \, dx - \int_0^{\frac{\pi}{2}} \log 2 \, dx$$

put, $2x = t \Rightarrow 2dx = dt$

When $x = 0, t = 0$

$$\therefore I = \frac{1}{2} \frac{\pi}{\int_0^a} \log \sin t \, dt - \frac{\pi}{2} \log 2$$

$$\Rightarrow I = \frac{1}{2} - \frac{\pi}{2} \log 2$$

$$\Rightarrow I = -\pi \log 2$$

Question 17:

$$\int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}} dx$$

Solution:

Let $I = \int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}} dx$ ...(1)

We know that, $\left(\int_0^a f(x) \, dx = \int_0^a f(a-x) \, dx \right)$

$$I = \int \frac{\sqrt{a-x}}{\sqrt{a-x} + \sqrt{x}} dx \quad \cdots (2)$$

Adding (1) and (2), we get

$$2I = \int_0^a \frac{\sqrt{x} + \sqrt{a-x}}{\sqrt{x} + \sqrt{a-x}} dx$$

$$\Rightarrow 2I = \int_0^a 1.dx$$

$$\Rightarrow 2I = [x]_0^a$$

$$\Rightarrow 2I = a$$

$$\Rightarrow I = \frac{a}{2}$$

Question 18:

$\int_0^4 |x-1| dx$

Solution:

$\int_0^4 |x-1| dx$

Since,

$(x-1) \le 0$ when $0 \le x \le 1$ and $(x-1) \ge 0$ when $1 \le x \le 4$

$I = \int_0^1 |x-1| dx + \int_1^4 |x-1| dx \qquad \left(\int_b^b f(x) dx = \int_b^c f(x) dx + \int_c^b f(x) dx \right)$

$I = \int_0^1 -(x-1) dx + \int_0^4 (x-1) dx$

$= \left[x - \dfrac{x^2}{2} \right]_0^1 + \left[\dfrac{x^2}{2} - x \right]_1^4 = 1 - \dfrac{1}{2} + \dfrac{(4)^2}{2} - 4 - \dfrac{1}{2} + 1$

$= 1 - \dfrac{1}{2} + 8 - 4 - \dfrac{1}{2} + 1$

$= 5$

Question 19:

Show that $\int_0^a f(x)g(x) dx = 2\int_0^a f(x) dx$ if f and g are defined as $f(x) = f(a-x)$ and $g(x) = (a-x) = 4$

Solution:

Let

$I = \int_0^a f(x)g(x) dx \quad \dots(1)$

$\Rightarrow \int_0^a f(a-x)g(a-x) dx \qquad \left(\int_0^a f(x) dx = \int_0^a f(a-x) dx \right)$

$\Rightarrow \int_0^a f(x)g(a-x) dx \quad \dots(2)$

Adding (1) and (2), we get

$2I = \int_0^a \{ f(x)g(x) + f(x)g(a-x) \} dx$

$\Rightarrow 2I = \int_0^a f(x)\{ g(x) + g(a-x) \} dx$

$\Rightarrow 2I = \int_0^a f(x) \times 4 dx \qquad [g(x) + g(a-x) = 4]$

$\Rightarrow I = 2\int_0^a f(x) dx$

Question 20:

The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(x^3 + x\cos x + \tan^5 x + 1\right)dx$ is

A. 0
B. 2
C. π
D. 1

Solution:

Consider, $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(x^3 + x\cos x + \tan^5 x + 1\right)dx$

$\Rightarrow I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 \, dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos x \, dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan^5 x \, dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 . dx$

For $f(x)$ an even function, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

If $f(x)$ is an odd function, then $\int_{-a}^{a} f(x)\,dx$
And

$I = 0 + 0 + 0 + 2\int_{0}^{\frac{\pi}{2}} 1 . dx$

$= 2\left[x\right]_{0}^{\frac{\pi}{2}}$

$= \frac{2\pi}{2}$

$= \pi$

Thus, the correct is option C.

Question 21:

The value of $\int_0^{\frac{\pi}{2}} \left(\frac{4+3\sin x}{4+3\cos x} \right) dx$ is

A. 2

B. $\frac{3}{4}$

C. 0

D. -2

Solution:

Let $I = \int_0^{\frac{\pi}{2}} \left(\frac{4+3\sin x}{4+3\cos x} \right) dx$ ...(1)

$\Rightarrow I = I = \int_0^{\frac{\pi}{2}} \left(\frac{4+3\sin\left(\frac{\pi}{2}-x\right)}{4+3\cos\left(\frac{\pi}{2}-x\right)} \right) dx \qquad \left(\int_0^a f(x) dx = \int_0^a f(a-x) dx \right)$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\cos x}{4+3\sin x} \right)$ ...(2)

Adding (1) and (2), we get

$2I = \int_0^{\frac{\pi}{2}} \left\{ \log\left(\frac{4+3\sin x}{4+3\cos x} \right) + \log\left(\frac{4+3\cos x}{4+3\sin x} \right) \right\} dx$

$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \left(\frac{4+3\sin x}{4+3\cos x} \times \frac{4+3\cos x}{4+3\sin x} \right) dx$

$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \log 1 dx$

$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 0 dx$

$\Rightarrow I = 0$

Thus, the correct option is C.

MISCELLANEOUS EXERCISE

Integrate the functions in Exercises 1 to 24.

Question 1:

$$\frac{1}{x - x^3}$$

Solution:

$$\frac{1}{x - x^3} = \frac{1}{x\left(1 - x^2\right)} = \frac{1}{x(1-x)(1+x)}$$

Let $\dfrac{1}{x(1-x)(1+x)} = \dfrac{A}{x} + \dfrac{B}{(1-x)} + \dfrac{C}{(1+x)}$ ...(1)

$$\Rightarrow 1 = A\left(1 - x^2\right) + Bx(1+x) + Cx(1-x)$$

$$\Rightarrow 1 = A - Ax^2 + Bx + Bx^2 + Cx - Cx^2$$

Equating the coefficients of x^2, x and constant terms, we get

$$-A + B - C = 0$$

$$B + C = 0$$

$$A = 1$$

On solving these equations, we get

$$A = 1$$

$$B = \frac{1}{2}$$

$$C = -\frac{1}{2}$$

From equation (1), we get

$$\frac{1}{x(1-x)(1+x)} = \frac{1}{x} + \frac{1}{2(1-x)} - \frac{1}{2(1+x)}$$

$$\Rightarrow \int \frac{1}{x(1-x)(1+x)} dx = \int \frac{1}{x} dx + \frac{1}{2}\int \frac{1}{(1-x)} dx - \frac{1}{2}\int \frac{1}{(1+x)} dx$$

$$= \log|x| - \frac{1}{2}\log|(1-x)| - \frac{1}{2}\log|(1+x)|$$

$$= \log|x| - \log\left|(1-x)^{\frac{1}{2}}\right| - \log\left|(1+x)^{\frac{1}{2}}\right| = \log\left|\frac{x}{(1-x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}}}\right| + C$$

$$= \log\left|\left(\frac{x^2}{1-x^2}\right)^{\frac{1}{2}}\right| + C = \frac{1}{2}\log\left|\frac{x^2}{1-x^2}\right| + C$$

Question 2:

$$\frac{1}{\sqrt{x+a}+\sqrt{x+b}}$$

Solution:

$$\frac{1}{\sqrt{x+a}+\sqrt{x+b}}=\frac{1}{\sqrt{x+a}+\sqrt{x+b}}\times\frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{x+a}-\sqrt{x+b}}$$

$$=\frac{\sqrt{x+a}-\sqrt{x+b}}{(x+a)-(x+b)}=\frac{\left(\sqrt{x+a}-\sqrt{x+b}\right)}{a-b}$$

$$\Rightarrow\int\frac{1}{\sqrt{x+a}+\sqrt{x+b}}dx=\frac{1}{a-b}\int\left(\sqrt{x+a}-\sqrt{x+b}\right)dx$$

$$=\frac{1}{(a-b)}\left[\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}}-\frac{(x+b)^{\frac{3}{2}}}{\frac{3}{2}}\right]=\frac{2}{3(a-b)}\left[(x+a)^{\frac{3}{2}}-(x+b)^{\frac{3}{2}}\right]+C$$

Question 3:

$$\frac{1}{x\sqrt{ax-x^2}}\qquad\left[\text{Hint: }x=\frac{a}{t}\right]$$

Solution:

$$\frac{1}{x\sqrt{ax-x^2}}$$

Let $x=\dfrac{a}{t}\Rightarrow dx=-\dfrac{a}{t^2}dt$

$$\Rightarrow\int\frac{1}{x\sqrt{ax-x^2}}dx=\int\frac{1}{\dfrac{a}{t}\sqrt{a.\dfrac{a}{t}-\left(\dfrac{a}{t}\right)^2}}\left(-\frac{a}{t^2}dt\right)$$

$$=-\int\frac{1}{at}\frac{1}{\sqrt{\dfrac{1}{t}-\dfrac{1}{t^2}}}dt=-\frac{1}{a}\int\frac{1}{\sqrt{\dfrac{t^2}{t}-\dfrac{t^2}{t^2}}}dt$$

$$=-\frac{1}{a}\int\frac{1}{\sqrt{t-1}}dt$$

$$=-\frac{1}{a}\left[2\sqrt{t-1}\right]+C$$

$$=-\frac{1}{a}\left[2\sqrt{\frac{a}{x}-1}\right]+C$$

$$=-\frac{2}{a}\left(\sqrt{\frac{a-x}{x}}\right)+C$$

Question 4:

$$\frac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}$$

Solution:

$$\frac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}$$

Multiplying and dividing by x^{-3}, we get

$$\frac{x^{-3}}{x^2 x^{-3}\left(x^4+1\right)^{\frac{3}{4}}} = \frac{x^{-3}\left(x^4+1\right)^{\frac{-3}{4}}}{x^2 x^{-3}}$$

$$= \frac{\left(x^4+1\right)^{\frac{-3}{4}}}{x^5\left(x^4\right)^{\frac{-3}{4}}} = \frac{1}{x^5}\left(\frac{x^4+1}{x^4}\right)^{\frac{-3}{4}}$$

$$= \frac{1}{x^5}\left(1+\frac{1}{x^4}\right)^{\frac{-3}{4}}$$

Let

$$\frac{1}{x^4} = t \Rightarrow -\frac{4}{x^5}dx = dt \Rightarrow \frac{1}{x^5}dx = -\frac{dt}{4}$$

$$\therefore \int \frac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}dx = \int \frac{1}{x^5}\left(1+\frac{1}{x^4}\right)^{\frac{-3}{4}}dx = -\frac{1}{4}\int\left(1+t\right)^{\frac{-3}{4}}dt$$

$$= -\frac{1}{4}\left[\frac{\left(1+t\right)^{\frac{1}{4}}}{\frac{1}{4}}\right] + C = -\frac{1}{4}\frac{\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}}}{\frac{1}{4}} + C$$

$$= -\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}} + C$$

Question 5:

$$\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}} \qquad \left[\text{Hint: } \frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}} = \frac{1}{x^{\frac{1}{3}}\left(1+x^{\frac{1}{6}}\right)} \text{ put } x=t^6 \right]$$

Solution:

$$\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}} = \frac{1}{x^{\frac{1}{3}}\left(1+x^{\frac{1}{6}}\right)}$$

Let $x=t^6 \Rightarrow dx=6t^5 dt$

$$\therefore \int \frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}\,dx = \int \frac{1}{x^{\frac{1}{3}}\left(1+x^{\frac{1}{6}}\right)}\,dx = \int \frac{6t^5}{t^2(1+t)}\,dt$$

$$= 6\int \frac{t^3}{(1+t)}\,dt$$

Adding and Substracting 1 in Numerator

$$= 6\int \frac{t^3+1-1}{1+t}\,dt$$

$$= 6\int \left(\frac{t^3+1}{1+t} - \frac{1}{1+t}\right)dt$$

Using $a^3+b^3 = (a+b)\left(a^2+b^2-ab\right)$

$$= 6\int \left\{ \frac{(1+t)\left(t^2+1^2-1\times t\right)}{1+t} - \frac{1}{1+t}\right\}dt$$

$$= 6\int \left\{ \left(t^2-t+1\right) - \frac{1}{1+t}\right\}dt$$

$$= 6\left[\left(\frac{t^3}{3}\right) - \left(\frac{t^2}{2}\right) + t - \log|1+t|\right]$$

$$= 2x^{\frac{1}{2}} - 3x^{\frac{1}{3}} + 6x^{\frac{1}{6}} - 6\log\left(1+x^{\frac{1}{6}}\right) + C$$

$$= 2\sqrt{x} - 3x^{\frac{1}{3}} + 6x^{\frac{1}{6}} - 6\log\left(1+x^{\frac{1}{6}}\right) + C$$

Question 6:

$$\frac{5x}{(x+1)(x^2+9)}$$

Solution:

Consider, $\dfrac{5x}{(x+1)(x^2+9)} = \dfrac{A}{(x+1)} + \dfrac{Bx+C}{(x^2+9)}$ ...(1)

$\Rightarrow 5x = A(x^2+9) + (Bx+C)(x+1)$

$\Rightarrow 5x = Ax^2 + 9A + Bx^2 + Bx + Cx + C$

Equating the coefficients of x^2, x and constant term, we get

$A + B = 0$

$B + C = 5$

$9A + C = 0$

On solving these equations, we get

$A = -\dfrac{1}{2}$

$B = \dfrac{1}{2}$

$C = \dfrac{9}{2}$

From equation (1), we get

$$\frac{5x}{(x+1)(x^2+9)} = \frac{-1}{2(x+1)} + \frac{\dfrac{x}{2}+\dfrac{9}{2}}{(x^2+9)}$$

$$\int \frac{5x}{(x+1)(x^2+9)}dx = \int \left\{ \frac{-1}{2(x+1)} + \frac{(x+9)}{2(x^2+9)} \right\}dx$$

$$= -\frac{1}{2}\log|x+1| + \frac{1}{2}\int \frac{x}{x^2+9}dx + \frac{9}{2}\int \frac{1}{x^2+9}dx = -\frac{1}{2}\log|x+1| + \frac{1}{4}\int \frac{2x}{x^2+9}dx + \frac{9}{2}\int \frac{1}{x^2+9}dx$$

$$= -\frac{1}{2}\log|x+1| + \frac{1}{4}\log|x^2+9| + \frac{9}{2}\cdot\frac{1}{3}\tan^{-1}\frac{x}{3} + C$$

$$= -\frac{1}{2}\log|x+1| + \frac{1}{4}\log(x^2+9) + \frac{3}{2}\tan^{-1}\frac{x}{3} + C$$

Question 7:

$$\frac{\sin x}{\sin (x-a)}$$

Solution:

$$\frac{\sin x}{\sin (x-a)}$$

Put, $x - a = t \Rightarrow dx = dt$

$$\int \frac{\sin x}{\sin (x-a)}dx = \int \frac{\sin (t+a)}{\sin t}dt$$

$$= \int \frac{\sin t \cos a + \cos t \sin a}{\sin t}dt = \int (\cos a + \cot t \sin a)dt$$

$$= t \cos a + \sin a \log|\sin t| + C_1$$

$$= (x-a)\cos a + \sin a \log|\sin (x-a)| + C_1$$

$$= x\cos a + \sin a \log|\sin (x-a)| - a\cos a + C_1$$

$$= \sin a \log|\sin (x-a)| + x\cos a + C$$

Question 8:

$$\frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}}$$

Solution:

$$\frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}} = \frac{e^{4\log x}\left(e^{\log x} - 1\right)}{e^{2\log x}\left(e^{\log x} - 1\right)}$$

$$= e^{2\log x}$$

$$= e^{\log x^2}$$

$$= x^2$$

$$\therefore \int \frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}}\,dx = \int x^2 dx = \frac{x^3}{3} + C$$

Question 9:

$$\frac{\cos x}{\sqrt{4 - \sin^2 x}}$$

Solution:

$$\frac{\cos x}{\sqrt{4 - \sin^2 x}}$$

Put, $\sin x = t \Rightarrow \cos x\,dx = dt$

$$\Rightarrow \int \frac{\cos x}{\sqrt{4 - \sin^2 x}}dx = \int \frac{dt}{\sqrt{(2)^2 - (t)^2}}$$

$$= \sin^{-1}\left(\frac{t}{2}\right) + C$$

$$= \sin^{-1}\left(\frac{\sin x}{2}\right) + C$$

$$= \frac{x}{2} + C$$

Question 10:

$$\frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x}$$

Solution:

$$\frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x} = \frac{\left(\sin^4 x - \cos^4 x\right)\left(\sin^4 x + \cos^4 x\right)}{\sin^2 x + \cos^2 x - \sin^2 x \cos^2 x - \sin^2 x \cos^2 x}$$

$$= \frac{\left(\sin^4 x + \cos^4 x\right)\left(\sin^2 x - \cos^2 x\right)\left(\sin^2 x + \cos^2 x\right)}{\left(\sin^2 x - \sin^2 x \cos^2 x\right) + \left(\cos^2 x - \sin^2 x \cos^2 x\right)}$$

$$= \frac{\left(\sin^4 x + \cos^4 x\right)\left(\sin^2 x - \cos^2 x\right)}{\sin^2 x \left(1 - \cos^2 x\right) + \cos^2 x \left(1 - \sin^2 x\right)}$$

$$= \frac{-\left(\sin^4 x + \cos^4 x\right)\left(\cos^2 x - \sin^2 x\right)}{\left(\sin^4 x + \cos^4 x\right)}$$

$$= -\cos 2x$$

$$\therefore \int \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x} dx = \int -\cos 2x\, dx = -\frac{\sin 2x}{2} + C$$

Question 11:

$$\frac{1}{\cos(x+a)\cos(x+b)}$$

Solution:

$$\frac{1}{\cos(x+a)\cos(x+b)}$$

Multiplying and dividing by $\sin(a-b)$, we get

$$\frac{1}{\sin(a-b)}\left[\frac{\sin(a-b)}{\cos(x+a)\cos(x+b)}\right]$$

$$=\frac{1}{\sin(a-b)}\left[\frac{\sin[(x+a)-(x+b)]}{\cos(x+a)\cos(x+b)}\right]$$

$$=\frac{1}{\sin(a-b)}\left[\frac{\sin(x+a)\cos(x+b)-\cos(x+a)\sin(x+b)}{\cos(x+a)\cos(x+b)}\right]$$

$$=\frac{1}{\sin(a-b)}\left[\frac{\sin(x+a)}{\cos(x+a)}-\frac{\sin(x+b)}{\cos(x+b)}\right]$$

$$=\frac{1}{\sin(a-b)}\left[\tan(x+a)-\tan(x+b)\right]$$

$$\int\frac{1}{\cos(x+a)\cos(x+b)}dx=\frac{1}{\sin(a-b)}\int\left[\tan(x+a)-\tan(x+b)\right]dx$$

$$=\frac{1}{\sin(a-b)}\left[-\log|\cos(x+a)|+\log|\cos(x+b)|\right]+C$$

$$=\frac{1}{\sin(a-b)}\log\left|\frac{\cos(x+b)}{\cos(x+a)}\right|+C$$

Question 12:

$$\frac{x^3}{\sqrt{1-x^8}}$$

Solution:

$$\frac{x^3}{\sqrt{1-x^8}}$$

Put, $x^4 = t \Rightarrow 4x^3 dx = dt$

$$\Rightarrow \int\frac{x^3}{\sqrt{1-x^8}}dx = \frac{1}{4}\int\frac{dt}{\sqrt{1-t^2}}$$

$$=\frac{1}{4}\sin^{-1}t+C$$

$$=\frac{1}{4}\sin^{-1}(x^4)+C$$

$$\frac{e^x}{\left(1+e^x\right)\left(2+e^x\right)}$$

Solution:

$$\frac{e^x}{\left(1+e^x\right)\left(2+e^x\right)}$$

Put $e^x = t \Rightarrow e^x dx = dt$

$$\Rightarrow \int \frac{e^x}{\left(1+e^x\right)\left(2+e^x\right)} dx = \int \frac{dt}{\left(t+1\right)\left(t+2\right)}$$

$$= \int \left[\frac{1}{\left(t+1\right)} - \frac{1}{\left(t+2\right)}\right] dt$$

$$= \log|t+1| - \log|t+2| + C$$

$$= \log \left|\frac{t+1}{t+2}\right| + C$$

$$= \log \left|\frac{1+e^x}{2+e^x}\right| + C$$

Question 14:

$$\frac{1}{\left(x^2+1\right)\left(x^2+4\right)}$$

Solution:

$$\therefore \frac{1}{\left(x^2+1\right)\left(x^2+4\right)} = \frac{Ax+B}{\left(x^2+1\right)} + \frac{Cx+D}{\left(x^2+4\right)}$$

$$\Rightarrow 1 = \left(Ax+B\right)\left(x^2+4\right) + \left(Cx+D\right)\left(x^2+1\right)$$

$$\Rightarrow 1 = Ax^3 + 4Ax + Bx^2 + 4B + Cx^3 + Cx + Dx^2 + D$$

Equating the coefficients of x^3, x^2, x and constant term, we get

$A + C = 0$

$B + D = 0$

$4A + C = 0$

$4B + D = 1$

On solving these equations, we get

$A = 0$

$B = \dfrac{1}{3}$

$C = 0$

$D = -\dfrac{1}{3}$

From equation (1), we get

$$\frac{1}{\left(x^2+1\right)\left(x^2+4\right)} = \frac{1}{3\left(x^2+1\right)} - \frac{1}{3\left(x^2+4\right)}$$

$$\int \frac{1}{\left(x^2+1\right)\left(x^2+4\right)} dx = \frac{1}{3}\int \frac{1}{x^2+1} dx - \frac{1}{3}\int \frac{1}{x^2+4} dx$$

$$= \frac{1}{3}\tan^{-1} x - \frac{1}{3}\cdot\frac{1}{2}\tan^{-1}\frac{x}{2} + C$$

$$= \frac{1}{3}\tan^{-1} x - \frac{1}{6}\tan^{-1}\frac{x}{2} + C$$

Question 15:

$\cos^3 x e^{\log \sin x}$

Solution:

$$\cos^3 x e^{\log \sin x} = \cos^3 x \times \sin x$$

Let $\cos x = t \Rightarrow -\sin x\, dx = dt$

$$\Rightarrow \int \cos^3 x e^{\log \sin x} dx = \int \cos^3 x \sin x\, dx$$

$$= -\int t^3 dt$$

$$= -\frac{t^4}{4} + C$$

$$= -\frac{\cos^4 x}{4} + C$$

Question 16:

$e^{3\log x}\left(x^4+1\right)^{-1}$

Solution:

$$e^{3\log x}\left(x^4+1\right)^{-1} = e^{\log x^3}\left(x^4+1\right)^{-1} = \frac{x^3}{\left(x^4+1\right)}$$

Let $x^4 + 1 = t \Rightarrow 4x^3 dx = dt$

$$\Rightarrow \int e^{3\log x}\left(x^4+1\right)^{-1}dx = \int \frac{x^3}{\left(x^4+1\right)}dx$$

$$= \frac{1}{4}\int \frac{dt}{t}$$

$$= \frac{1}{4}\log|t| + C$$

$$= \frac{1}{4}\log|x^4+1| + C$$

$$= \frac{1}{4}\log\left(x^4+1\right) + C$$

Question 17:

$$f'(ax+b)\left[f(ax+b)\right]^n$$

Solution:

$$f'(ax+b)\left[f(ax+b)\right]^n$$

Put, $f(ax+b) = t \Rightarrow af'(ax+b)dx = dt$

$$\Rightarrow \int f'(ax+b)\left[f(ax+b)\right]^n dx = \frac{1}{a}\int t^n dt$$

$$= \frac{1}{a}\left[\frac{t^{n+1}}{n+1}\right] = \frac{1}{a(n+1)}\left(f(ax+b)\right)^{n+1} + C$$

Question 18:

$$\frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}}$$

Solution:

$$\frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}} = \frac{1}{\sqrt{\sin^3 x (\sin x \cos\alpha + \cos x \sin\alpha)}}$$

$$= \frac{1}{\sqrt{\sin^4 x \cos\alpha + \sin^3 x \cos x \sin\alpha}}$$

$$= \frac{1}{\sin^2 x\sqrt{\cos\alpha + \cot x \sin\alpha}} = \frac{\cos ec^2 x}{\sqrt{\cos\alpha + \cot x \sin\alpha}}$$

Put, $\cos\alpha + \cot x \sin\alpha = t \Rightarrow -\cos ec^2 x \sin\alpha\, dx = dt$

$$\therefore \int \frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}} dx = \int \frac{\cos ec^2 x}{\sqrt{\cos\alpha + \cot x \sin\alpha}} dx$$

$$= \frac{-1}{\sin\alpha} \int \frac{dt}{\sqrt{t}}$$

$$= \frac{-1}{\sin\alpha} \left[2\sqrt{t} \right] + C$$

$$= \frac{-1}{\sin\alpha} \left[2\sqrt{\cos\alpha + \cot x \sin\alpha} \right] + C$$

$$= \frac{-2}{\sin\alpha} \sqrt{\cos\alpha + \frac{\cos x \sin\alpha}{\sin x}} + C$$

$$= \frac{-2}{\sin\alpha} \sqrt{\frac{\sin x \cos\alpha + \cos x \sin\alpha}{\sin x}} + C = \frac{-2}{\sin\alpha} \sqrt{\frac{\sin(x+\alpha)}{\sin x}} + C$$

Question 19:

$$\frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}, x \in [0,1]$$

Solution:

Let $I = \dfrac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}} dx$

As we know that, $\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x} = \dfrac{\pi}{2}$

$$\Rightarrow I = \int \frac{\left(\dfrac{\pi}{2} - \cos^{-1}\sqrt{x}\right) - \cos^{-1}\sqrt{x}}{\dfrac{\pi}{2}} dx$$

$$= \frac{2}{\pi} \int \left(\frac{\pi}{2} - 2\cos^{-1}\sqrt{x} \right) dx$$

$$= \frac{2}{\pi} \cdot \frac{\pi}{2} \int 1.dx - \frac{4}{\pi} \int \cos^{-1}\sqrt{x} dx$$

$$= x - \frac{4}{\pi} \int \cos^{-1}\sqrt{x} dx \qquad ...(1)$$

Let $I_1 = \int \cos^{-1}\sqrt{x} dx$

Also, let $\sqrt{x} = t \Rightarrow dx = 2t dt$

$$\Rightarrow I_1 = 2\int \cos^{-1} t . t \, dt$$

$$= 2\left[\cos^{-1} t . \frac{t^2}{2} - \int \frac{-1}{\sqrt{1-t^2}} . \frac{t^2}{2} dt\right]$$

$$= t^2 \cos^{-1} t + \int \frac{t^2}{\sqrt{1-t^2}} dt$$

$$= t^2 \cos^{-1} t - \int \frac{1-t^2-1}{\sqrt{1-t^2}} dt$$

$$= t^2 \cos^{-1} t - \int \sqrt{1-t^2} \, dt + \int \frac{1}{\sqrt{1-t^2}} dt$$

$$= t^2 \cos^{-1} t - \frac{1}{2}\sqrt{1-t^2} - \frac{1}{2}\sin^{-1} t + \sin^{-1} t$$

$$= t^2 \cos^{-1} t - \frac{1}{2}\sqrt{1-t^2} + \frac{1}{2}\sin^{-1} t$$

From equation (1), we get

$$I = x - \frac{4}{\pi}\left[t^2 \cos^{-1} t - \frac{t}{2}\sqrt{1-t^2} + \frac{1}{2}\sin^{-1} t\right]$$

$$= x - \frac{4}{\pi}\left[x \cos^{-1}\sqrt{x} - \frac{\sqrt{x}}{2}\sqrt{1-x} + \frac{1}{2}\sin^{-1}\sqrt{x}\right]$$

$$x - \frac{4}{\pi}\left[x\left(\frac{\pi}{2} - \sin^{-1}\sqrt{x}\right) - \frac{\sqrt{x-x^2}}{2} + \frac{\pi}{2}\sin^{-1}\sqrt{x}\right]$$

$$= x - 2x + \frac{4x}{\pi}\sin^{-1}\sqrt{x} + \frac{2}{\pi}\sqrt{x-x^2} - \frac{2}{\pi}\sin^{-1}\sqrt{x}$$

$$-x + \frac{2}{\pi}\left[(2x-1)\sin^{-1}\sqrt{x}\right] + \frac{2}{\pi}\sqrt{x-x^2} + C$$

$$= \frac{2(2x-1)}{\pi}\sin^{-}\sqrt{x} + \frac{2}{\pi}\sqrt{x-x^2} - x + C$$

Question 20:

$$\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$$

Solution:

$$I = \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$$

Put, $x = \cos^2\theta \Rightarrow dx = -2\sin\theta\cos\theta \, d\theta$

$$I = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\left(-2\sin\theta\cos\theta\right)d\theta = -\int\sqrt{\frac{2\sin^2\dfrac{\theta}{2}}{2\cos^2\dfrac{\theta}{2}}}\,\sin 2\theta\,d\theta$$

$$= -2\int\frac{\sin\dfrac{\theta}{2}}{\cos\dfrac{\theta}{2}}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)\cos\theta\,d\theta$$

$$= -4\int\sin^2\frac{\theta}{2}\cos\theta\,d\theta$$

$$= -4\int\sin^2\frac{\theta}{2}\left(2\cos^2\frac{\theta}{2}-1\right)d\theta$$

$$= -4\int\left(2\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\right)d\theta$$

$$= -8\int\sin^2\frac{\theta}{2}.\cos^2\frac{\theta}{2}\,d\theta + 4\int\sin^2\frac{\theta}{2}d\theta$$

$$= -2\int\sin^2\frac{\theta}{2}d\theta + 4\int\sin^2\frac{\theta}{2}d\theta$$

$$= -2\int\left(\frac{1-\cos 2\theta}{2}\right)d\theta + 4\int\frac{1-\cos\theta}{2}d\theta$$

$$= -2\left[\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right] + 4\left[\frac{\theta}{2}-\frac{\sin\theta}{2}\right] + C$$

$$= -\theta + \frac{\sin 2\theta}{2} + 2\theta - 2\sin\theta + C$$

$$= \theta + \frac{\sin 2\theta}{2} + 2\sin\theta + C$$

$$= \theta + \frac{2\sin\theta\cos\theta}{2} - 2\sin\theta + C$$

$$= \theta + \sqrt{1-\cos^2\theta}.\cos\theta - 2\sqrt{1-\cos^2\theta} + C$$

$$= \cos^{-1}\sqrt{x} + \sqrt{1-x}.\sqrt{x} - 2\sqrt{1-x} + C$$

$$= -2\sqrt{1-x} + \cos^{-1}\sqrt{x} + \sqrt{x(1-x)} + C$$

$$= -2\sqrt{1-x} + \cos^{-1}\sqrt{x} + \sqrt{x-x^2} + C$$

Question 21:

$$\frac{2+\sin 2x}{1+\cos 2x}e^x$$

Solution:

$$I = \int \left(\frac{2+\sin 2x}{1+\cos 2x} \right) e^x$$

$$= \int \left(\frac{2+2\sin x \cos x}{2\cos^2 x} \right) e^x$$

$$= \int \left(\frac{1+\sin x \cos x}{\cos^2 x} \right) e^x$$

$$= \int \left(\sec^2 x + \tan x \right) e^x$$

Let $f(x) = \tan x \Rightarrow f'(x) = \sec^2 x$

$$\therefore I = \int \left(f(x) + f'(x) \right) e^x dx$$

$$= e^x f(x) + C$$

$$= e^x \tan x + C$$

Question 22:

$$\frac{x^2+x+1}{(x+1)^2(x+2)}$$

Solution:

Let $\dfrac{x^2+x+1}{(x+1)^2(x+2)} = \dfrac{A}{(x+1)} + \dfrac{B}{(x+1)^2} + \dfrac{C}{(x+2)}$ $\quad \cdots (1)$

$$\Rightarrow x^2 + x + 1 = A(x+1)(x+2) + B(x+2) + C(x^2+2x+1)$$

$$\Rightarrow x^2 + x + 1 = A(x^2+3x+2) + B(x+2) + C(x^2+2x+1)$$

$$\Rightarrow x^2 + x + 1 = (A+C)x^2 + (3A+B+2C)x + (2A+2B+C)$$

Equating the coefficients of x^2, x and constant term, we get

$A + C = 1$

$3A + B + 2C = 1$

$2A + 2B + C = 1$

On solving these equations, we get

$A = -2$

$B = 1$

$C = 3$

From equation (1), we get

$$\frac{x^2+x+1}{(x+1)^2(x+2)}=\frac{-2}{(x+1)}+\frac{3}{(x+2)}+\frac{1}{(x+1)^2}$$

$$\int\frac{x^2+x+1}{(x+1)^2(x+2)}dx=-2\int\frac{1}{x+1}dx+3\int\frac{1}{(x+2)}dx+\int\frac{1}{(x+1)^2}dx$$

$$=-2\log|x+1|+3\log|x+2|-\frac{1}{(x+1)}+C$$

Question 23:

$$\tan^{-1}\sqrt{\frac{1-x}{1+x}}$$

Solution:

$$I=\tan^{-1}\sqrt{\frac{1-x}{1+x}}dx$$

Let $x=\cos\theta\Rightarrow dx=-\sin\theta d\theta$

$$I=\int\tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}(-\sin\theta)d\theta$$

$$=-\int\tan^{-1}\sqrt{\frac{2\sin^2\dfrac{\theta}{2}}{2\cos^2\dfrac{\theta}{2}}}\sin\theta d\theta=-\int\tan^{-1}\tan\frac{\theta}{2}\sin\theta d\theta$$

$$=-\frac{1}{2}\int\theta.\sin\theta d\theta=-\frac{1}{2}\Big[\theta.(-\cos\theta)-\int1.(-\cos\theta)d\theta\Big]$$

$$=-\frac{1}{2}\Big[-\theta\cos\theta+\sin\theta\Big]$$

$$=\frac{1}{2}\theta\cos\theta-\frac{1}{2}\sin\theta$$

$$=\frac{1}{2}\cos^{-1}x.x-\frac{1}{2}\sqrt{1-x^2}+C=\frac{x}{2}\cos^{-1}x-\frac{1}{2}\sqrt{1-x^2}+C$$

$$=\frac{1}{2}\left(x\cos^{-1}x-\sqrt{1-x^2}\right)+C$$

Question 24:

$$\frac{\sqrt{x^2+1}\left[\log\left(x^2+1\right)-2\log x\right]}{x^4}$$

Solution:

$$\frac{\sqrt{x^2+1}\left[\log\left(x^2+1\right)-2\log x\right]}{x^4}=\frac{\sqrt{x^2+1}}{x^4}\left[\log\left(x^2+1\right)-\log x^2\right]$$

$$=\frac{\sqrt{x^2+1}}{x^4}\left[\log\left(\frac{x^2+1}{x^2}\right)\right]$$

$$=\frac{\sqrt{x^2+1}}{x^4}\log\left(1+\frac{1}{x^2}\right)$$

$$=\frac{1}{x^3}\sqrt{\frac{x^2+1}{x^2}}\log\left(1+\frac{1}{x^2}\right)$$

$$=\frac{1}{x^3}\sqrt{1+\frac{1}{x^2}}\log\left(1+\frac{1}{x^2}\right)$$

Let $1+\dfrac{1}{x^2}=t\Rightarrow\dfrac{-2}{x^3}dx=dt$

$$\therefore I=\int\frac{1}{x^3}\sqrt{1+\frac{1}{x^2}}\log\left(1+\frac{1}{x^2}\right)dx$$

$$=-\frac{1}{2}\int\sqrt{t}\log t\,dt=-\frac{1}{2}\int t^{\frac{1}{2}}\log t\,dt$$

Using integration by parts, we get

$$I=-\frac{1}{2}\left[\log t.\int t^{\frac{1}{2}}dt-\left\{\left(\frac{d}{dt}\log t\right)\int t^{\frac{1}{2}}dt\right\}dt\right]$$

$$=-\frac{1}{2}\left[\log t.\frac{t^{\frac{3}{2}}}{\frac{3}{2}}-\int\frac{1}{t}.\frac{t^{\frac{3}{2}}}{\frac{3}{2}}dt\right]$$

$$=-\frac{1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\log t-\frac{2}{3}\int t^{\frac{1}{2}}dt\right]$$

$$=-\frac{1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\log t-\frac{4}{9}t^{\frac{3}{2}}\right]$$

$$=-\frac{1}{3}t^{\frac{3}{2}}\log t+\frac{2}{9}t^{\frac{3}{2}}$$

$$=-\frac{1}{3}t^{\frac{3}{2}}\left[\log t-\frac{2}{3}\right]$$

$$=-\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{\frac{3}{2}}\left[\log\left(1+\frac{1}{x^2}\right)-\frac{2}{3}\right]+C$$

Question 25:

$$\int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{1 - \sin x}{1 - \cos x} \right) dx$$

Solution:

$$I = \int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{1 - \sin x}{1 - \cos x} \right) dx$$

$$= \int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{1 - 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}} \right) dx = \int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{\cos ec^2\frac{x}{2}}{2} - \cot\frac{x}{2} \right) dx$$

Let $f(x) = -\cot\frac{x}{2}$

$$\Rightarrow f'(x) = -\left(\frac{1}{2}\cos ec^2\frac{x}{2} \right) = \frac{1}{2}\cos ec^2\frac{x}{2}$$

$$\therefore I = \int_{\frac{\pi}{2}}^{\pi} e^x \left(f(x) + f'(x) \right) dx$$

$$= \left[e^x f(x) dx \right]_{\frac{\pi}{2}}^{\pi}$$

$$= -\left[e^x \cot\frac{x}{2} \right]_{\frac{\pi}{2}}^{\pi}$$

$$= -\left[e^x \times \cot\frac{\pi}{2} - e^{\frac{\pi}{2}} \times \cot\frac{\pi}{4} \right]$$

$$= -\left[e^\pi \times 0 - e^{\frac{\pi}{2}} \times 1 \right]$$

$$= e^{\frac{\pi}{2}}$$

Question 26:

$$\int_{0}^{\frac{\pi}{4}} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x} dx$$

Solution:

Let $I = \int_{0}^{\frac{\pi}{4}} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x} dx$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \frac{\dfrac{(\sin x \cos x)}{\cos^4 x}}{\dfrac{(\cos^4 x + \sin^4 x)}{\cos^4 x}} dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \frac{\tan x \sec^2 x}{1 + \tan^4 x} dx$$

Put, $\tan^2 x = t \Rightarrow 2 \tan x \sec^2 x \, dx = dt$

When $x = 0, t = 0$ and when $x = \dfrac{\pi}{4}, t = 1$

$$\therefore I = \frac{1}{2} \int_0^1 \frac{dt}{1 + t^2} = \frac{1}{2} \left[\tan^{-1} t \right]_0^1$$

$$= \frac{1}{2} \left[\tan^{-1} 1 - \tan^{-1} 0 \right]$$

$$= \frac{1}{2} \left[\frac{\pi}{4} \right]$$

$$= \frac{\pi}{8}$$

Question 27:

$$\int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4 \sin^2 x} dx$$

Solution:

Consider, $I = \int_0^{\frac{\pi}{2}} \dfrac{\cos^2 x}{\cos^2 x + 4 \sin^2 x} dx$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4\left(1 - \cos^2 x\right)} dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4 - 4\cos^2 x} dx$$

$$\Rightarrow I = \frac{-1}{3} \int_0^{\frac{\pi}{2}} \frac{-3\cos^2 x}{4 - 3\cos^2 x} dx$$

$$\Rightarrow I = \frac{-1}{3} \int_0^{\frac{\pi}{2}} \frac{4 - 3\cos^2 x - 4}{4 - 3\cos^2 x} dx$$

$$\Rightarrow I = \frac{-1}{3} \int_0^{\frac{\pi}{2}} \frac{4 - 3\cos^2 x}{4 - 3\cos^2 x} dx + \frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{4}{4 - 3\cos^2 x} dx$$

$$\Rightarrow I = \frac{-1}{3} \int_0^{\frac{\pi}{2}} 1 \, dx + \frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{4\sec^2 x}{4\sec^2 x - 3} dx$$

$$\Rightarrow I = -\frac{1}{3} \left[x \right]_0^{\frac{\pi}{2}} + \frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{4\sec^2 x}{4\left(1 + \tan^2 x\right) - 3} dx$$

$$\Rightarrow I = -\frac{\pi}{6} + \frac{2}{3} \int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{1 + 4\tan^2 x} dx \qquad \ldots (1)$$

Consider, $\int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{1 + 4\tan^2 x} dx$

Put, $2\tan x = t \Rightarrow 2\sec^2 x \, dx = dt$

When $x = 0, t = 0$ and when $x = \dfrac{\pi}{2}, t = \infty$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{1 + 4\tan^2 x} dx = \int_0^{\infty} \frac{dt}{1 + t^2}$$

$$= \left[\tan^{-1} t \right]_0^{\infty}$$

$$= \left[\tan^{-1}(\infty) - \tan^{-1}(0) \right]$$

$$= \frac{\pi}{2}$$

Therefore, from (1), we get

$$I = -\frac{\pi}{6} + \frac{2}{3} \left[\frac{\pi}{2} \right] = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

Question 28:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$$

Solution:

Consider, $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$

$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{-(-\sin 2x)}} dx \Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{-(-1+1-2\sin\cos x)}} dx$

$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{1-(\sin x - \cos x)^2}} dx$

Let $(\sin x - \cos x) = t \Rightarrow (\sin x + \cos x) dx = dt$

When $x = \frac{\pi}{6}, t = \left(\frac{1-\sqrt{3}}{2}\right)$ and when $x = \frac{\pi}{3}, t = \left(\frac{\sqrt{3}-1}{2}\right)$

$I = \int_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$

$\Rightarrow I = \int_{-\left(\frac{1+\sqrt{3}}{2}\right)}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$

As $\frac{1}{\sqrt{1-(-t)^2}} = \frac{1}{\sqrt{1-t^2}}$, therefore, $\frac{1}{\sqrt{1-t^2}}$ is an even function

We know that if $f(x)$ is an even function, then $\int_{-a}^{a} f(x) dx = 2\int_{0}^{a} f(x) dx$

$\Rightarrow I = 2\int_{0}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$

$= \left[2\sin^{-1} t\right]_{0}^{\frac{\sqrt{3}-1}{2}}$

$= 2\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$

Question 29:

$$\int_0^1 \frac{dx}{\sqrt{1+x} - \sqrt{x}}$$

Solution:

Consider, $I = \int_0^1 \frac{dx}{\sqrt{1+x} - \sqrt{x}}$

$$I = \int_0^1 \frac{1}{\left(\sqrt{1+x} - \sqrt{x}\right)} \times \frac{\left(\sqrt{1+x} + \sqrt{x}\right)}{\left(\sqrt{1+x} + \sqrt{x}\right)} dx$$

$$= \int_0^1 \frac{\left(\sqrt{1+x} + \sqrt{x}\right)}{1+x-x} dx$$

$$= \int_0^1 \sqrt{1+x}\, dx + \int_0^1 \sqrt{x}\, dx$$

$$= \left[\frac{2}{3}(1+x)^{\frac{3}{2}}\right]_0^1 + \left[\frac{2}{3}(x)^{\frac{3}{2}}\right]_0^1$$

$$= \frac{2}{3}\left[(2)^{\frac{3}{2}} - 1\right] + \frac{2}{3}[1]$$

$$= \frac{2}{3}(2)^{\frac{3}{2}} = \frac{2.2\sqrt{2}}{3}$$

$$= \frac{4\sqrt{2}}{3}$$

Question 30:

$$\int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16\sin 2x}\, dx$$

Solution:

Consider, $I = \int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16\sin 2x}\, dx$

Put, $\sin x - \cos x = t \Rightarrow (\cos x + \sin x)\, dx = dt$

When $x = 0, t = -1$ and when $x = \frac{\pi}{4}, t = 0$

$\Rightarrow (\sin x - \cos x)^2 = t^2$

$\Rightarrow \sin^2 x + \cos^2 x - 2\sin x \cos x = t^2$

$\Rightarrow 1 - \sin 2x = t^2$

$\Rightarrow \sin 2x = 1 - t^2$

$\therefore I = \int_{-1}^{0} \dfrac{dt}{9 + 16(1 - t^2)}$

$= \int_{-1}^{0} \dfrac{dt}{9 + 16 - 16t^2}$

$= \int_{-1}^{0} \dfrac{dt}{25 - 16t^2} = \int_{-1}^{0} \dfrac{dt}{(5)^2 - (4t)^2}$

$= \dfrac{1}{4}\left[\dfrac{1}{2(5)} \log \left| \dfrac{5 + 4t}{5 - 4t} \right| \right]_{-1}^{0}$

$= \dfrac{1}{40}\left[\log(1) - \log \left| \dfrac{1}{9} \right| \right]$

$= \dfrac{1}{40} \log 9$

Question 31:

$\int_{0}^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x)\,dx$

Solution:

Consider, $I = \int_{0}^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x)\,dx = \int_{0}^{\frac{\pi}{2}} 2\sin x \cos x \tan^{-1}(\sin x)\,dx$

Put, $\sin x = t \Rightarrow \cos x\,dx = dt$

When $x = 0, t = 0$ and when $x = \dfrac{\pi}{2}, t = 1$

$\Rightarrow I = 2\int_{0}^{1} t\tan^{-1}(t)\,dt \qquad \ldots(1)$

Consider $\int t.\tan^{-1} t\,dt = \tan^{-1} t \int t\,dt - \int \left\{ \dfrac{d}{dt}(\tan^{-1} t) \int t\,dt \right\} dt$

$= \tan^{-1} t . \dfrac{t^2}{2} - \int \dfrac{1}{1 + t^2} . \dfrac{t^2}{2}\,dt$

$= \dfrac{t^2 \tan^{-1} t}{2} - \dfrac{1}{2} \int \dfrac{t^2 + 1 - 1}{1 + t^2}\,dt$

$= \dfrac{t^2 \tan^{-1} t}{2} - \dfrac{1}{2} \int 1.\,dt + \dfrac{1}{2} \int \dfrac{1}{1 + t^2}\,dt$

$= \dfrac{t^2 \tan^{-1} t}{2} - \dfrac{1}{2}t + \dfrac{1}{2} \tan^{-1} t$

From equation (1), we get

$$\Rightarrow 2\int_0^1 t.\tan^{-1} t \, dt = 2\left[\frac{t^2 \tan^{-1} t}{2} - \frac{t}{2} + \frac{1}{2}\tan^{-1} t\right]_0^1$$

$$= \left[\frac{\pi}{4} - 1 + \frac{\pi}{4}\right] = \frac{\pi}{2} - 1$$

Question 32:

$$\int_0^\pi \frac{x \tan x}{\sec x + \tan x} dx$$

Solution:

Let $\int_0^\pi \frac{x \tan x}{\sec x + \tan x} dx$ ...(1)

$$I = \int_0^\pi \left\{\frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x) + \tan(\pi - x)}\right\} dx \qquad \left(\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right)$$

$$\Rightarrow I = \int_0^\pi \left\{\frac{-(\pi - x)\tan x}{-(\sec x + \tan x)}\right\} dx$$

$$\Rightarrow I = \int_0^\pi \frac{(\pi - x)\tan x}{(\sec x + \tan x)} dx \qquad ...(2)$$

Adding (1) and (2), we get

$$2I = \int_0^\pi \frac{\pi \tan x}{\sec x + \tan x} dx \Rightarrow 2I = \pi \int_0^\pi \frac{\dfrac{\sin x}{\cos x}}{\dfrac{1}{\cos x} + \dfrac{\sin x}{\cos x}} dx$$

$$\Rightarrow 2I = \pi \int_0^\pi \frac{\sin x + 1 - 1}{1 + \sin x} dx$$

$$\Rightarrow 2I = \pi \int_0^\pi 1.dx - \pi \int_0^\pi \frac{1}{1 + \sin x} dx$$

$$\Rightarrow 2I = \pi \int_0^\pi 1.dx - \pi \int_0^\pi \frac{(1 - \sin x)}{(1 + \sin x)(1 - \sin x)} dx$$

$$\Rightarrow 2I = \pi [x]_0^\pi - \pi \int_0^\pi \frac{1 - \sin x}{\cos^2 x} dx$$

$$\Rightarrow 2I = \pi^2 - \pi \int_0^\pi (\sec^2 x - \tan x \sec x) dx$$

$$\Rightarrow 2I = \pi^2 - \pi [\tan x - \sec x]_0^\pi$$

$$\Rightarrow 2I = \pi^2 - \pi [\tan \pi - \sec \pi - \tan 0 + \sec 0]$$

$$\Rightarrow 2I = \pi^2 - \pi [0 - (-1) - 0 + 1]$$

$$\Rightarrow 2I = \pi^2 - 2\pi$$

$$\Rightarrow 2I = \pi (\pi - 2)$$

$$I = \frac{\pi}{2} (\pi - 2)$$

Question 33:

$$\int_1^4 \Big[|x-1| + |x-2| + |x-3| \Big] dx$$

Solution:

Consider, $I = \int_1^4 \Big[|x-1| + |x-2| + |x-3| \Big] dx$

$$\Rightarrow I = \int_1^4 |x-1| dx + \int_1^4 |x+2| dx + \int_1^4 |x+3| dx$$

$$I = I_1 + I_2 + I_3 \qquad \dots(1)$$

Where, $I_1 = \int_1^4 |x-1| dx, I_2 = \int_1^4 |x+2| dx$ and $I_3 = \int_1^4 |x+3| dx$

$$I_1 = \int_1^4 |x-1| dx$$

$(x-1) \geq 0$ *for* $1 \leq x \leq 4$

$\therefore I_1 = \int_1^4 (x-1) \, dx$

$\Rightarrow I_1 = \left[\dfrac{x^2}{2} - x \right]_1^4$

$\Rightarrow I_1 = \left[8 - 4 - \dfrac{1}{2} + 1 \right] = \dfrac{9}{2}$ ...(2)

$I_2 = \int_1^4 |x-2| \, dx$

$x - 2 \geq 0$ *for* $2 \leq x \leq 4$ and $x - 2 \leq 0$ *for* $1 \leq x \leq 2$

$\therefore I_2 = \int_1^2 (2-x) \, dx + \int_2^4 (x-2) \, dx$

$\Rightarrow I_2 = \left[2x - \dfrac{x^2}{2} \right]_1^2 + \left[\dfrac{x^2}{2} - 2x \right]_2^4 \Rightarrow I_2 = \left[4 - 2 - 2 + \dfrac{1}{2} \right] + [8 - 8 - 2 + 4]$

$\Rightarrow I_2 = \dfrac{1}{2} + 2 = \dfrac{5}{2}$

$\Rightarrow I_3 = \int_1^4 |x-3| \, dx$

$x - 3 \geq 0$ *for* $3 \leq x \leq 4$ and $x - 3 \leq 0$ *for* $1 \leq x \leq 2$

$\therefore I_3 = \int_1^3 (3-x) \, dx + \int_3^4 (x-3) \, dx$

$\Rightarrow I_3 = \left[3 - \dfrac{x^2}{2} \right]_1^3 + \left[\dfrac{x^2}{2} - 3x \right]_3^4$

$\Rightarrow I_3 = \left[9 - \dfrac{9}{2} - 3 + \dfrac{1}{2} \right] + \left[8 - 12 - \dfrac{9}{2} + 9 \right]$

$\Rightarrow I_3 = [6 - 4] + \left[\dfrac{1}{2} \right] = \dfrac{5}{2}$ ...(4)

From equations (1), (2), (3) and (4), we get

$I = \dfrac{9}{2} + \dfrac{5}{2} + \dfrac{5}{2} = \dfrac{19}{2}$

Question 34:

$\displaystyle\int_1^3 \dfrac{dx}{x^2(x+1)} = \dfrac{2}{3} + \log \dfrac{2}{3}$

Solution:

Consider, $\displaystyle\int_1^3 \dfrac{dx}{x^2(x+1)}$

Let, $\dfrac{1}{x^2(x+1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+1}$

$$\Rightarrow 1 = Ax(x+1) + B(x+1) + C(x^2)$$

$$\Rightarrow 1 = Ax^2 + Ax + Bx + B + Cx^2$$

Equating the coefficients of x^2, x and constant terms, we get

$$A + C = 0$$

$$A + B = 0$$

$$B = 1$$

On solving these equations, we get

$$A = -1$$

$$C = 1$$

$$B = 1$$

$$\therefore \frac{1}{x^2(x+1)} = \frac{-1}{x} + \frac{1}{x^2} + \frac{1}{(x+1)}$$

$$\Rightarrow I = \int_1^3 \left\{ -\frac{1}{x} + \frac{1}{x^2} + \frac{1}{(x+1)} \right\} dx = \left[-\log x - \frac{1}{x} + \log(x+1) \right]_1^3$$

$$= \left[\log\left(\frac{x+1}{x}\right) - \frac{1}{x} \right]_1^3 = \log\left(\frac{4}{3}\right) - \frac{1}{3} - \log\left(\frac{2}{1}\right) + 1$$

$$= \log 4 - \log 3 - \log 2 + \frac{2}{3}$$

$$= \log 2 - \log 3 + \frac{2}{3}$$

$$= \log\left(\frac{2}{3}\right) + \frac{2}{3}$$

Hence proved.

Question 35:

$$\int_0^1 xe^x dx = 1$$

Solution:

Let $I = \int_0^1 xe^x dx$

Using integration by parts, we get

$$I = x \int_0^1 e^x dx - \int_0^1 \left\{ \left(\frac{d}{dx}(x) \right) \int e^x dx \right\} dx$$

$$= \left[xe^x \right]_0^1 - \int_0^1 e^x dx$$

$$= \left[xe^x \right]_0^1 - \left[e^x \right]_0^1$$

$$= e - e + 1$$

$$= 1$$

Hence proved.

Question 36:

$$\int_{-1}^{1} x^{17} \cos^4 x\, dx = 0$$

Solution:

Consider, $I = \int_{-1}^{1} x^{17} \cos^4 x\, dx$

Let $f(x) = x^{17} \cos^4 x$

$\Rightarrow f(-x) = (-x)^{17} \cos^4(-x) = -x^{17} \cos^4 x = -f(x)$

$f(x)$ is an odd function.

We know that if $f(x)$ is an odd function, then $\int_{-a}^{a} f(x)\, dx = 0$

$\therefore I = \int_{-1}^{1} x^{17} \cos^4 x\, dx = 0$

Hence proved.

Question 37:

$$\int_{0}^{\frac{\pi}{2}} \sin^3 x\, dx = \frac{2}{3}$$

Solution:

Consider, $I = \int_{0}^{\frac{\pi}{2}} \sin^3 x\, dx$

$I = \int_{0}^{\frac{\pi}{2}} \sin^2 x \cdot \sin x\, dx$

$= \int_{0}^{\frac{\pi}{2}} \left(1 - \cos^2 x\right) \sin x\, dx$

$= \int_{0}^{\frac{\pi}{2}} \sin x\, dx - \int_{0}^{\frac{\pi}{2}} \cos^2 x \cdot \sin x\, dx$

$= \left[-\cos x\right]_{0}^{\frac{\pi}{2}} + \left[\dfrac{\cos^3 x}{3}\right]_{0}^{\frac{\pi}{2}}$

$= 1 + \dfrac{1}{3}[-1] = 1 - \dfrac{1}{3} = \dfrac{2}{3}$

Hence proved.

Question 38:

$$\int_0^{\frac{\pi}{4}} 2\tan^3 xdx = 1 - \log 2$$

Solution:

Consider, $I = \int_0^{\frac{\pi}{4}} 2\tan^3 xdx$

$I = \int_0^{\frac{\pi}{4}} 2\tan^2 x.\tan xdx = 2\int_0^{\frac{\pi}{4}} \left(\sec^2 - 1\right)\tan xdx$

$= 2\int_0^{\frac{\pi}{4}} \sec^2 x\tan xdx - 2\int_0^{\frac{\pi}{4}} \tan xdx$

$= 2\left[\dfrac{\tan^2 x}{2}\right]_0^{\frac{\pi}{4}} + 2\left[\log\cos x\right]_0^{\frac{\pi}{4}} = 1 + 2\left[\log\cos\dfrac{\pi}{4} - \log\cos 0\right]$

$= 1 + 2\left[\log\dfrac{1}{\sqrt{2}} - \log 1\right] = 1 - \log 2 - \log 1 = 1 - \log 2$

Hence proved.

Question 39:

$$\int_0^1 \sin^{-1} xdx = \dfrac{\pi}{2} - 1$$

Solution:

Let $\int_0^1 \sin^{-1} xdx$

$\Rightarrow I = \int_0^1 \sin^{-1} x.1.dx$

Using integration by parts, we get

$I = \left[\sin^{-1} x.x\right]_0^1 - \int_0^1 \dfrac{1}{\sqrt{1-x^2}} xdx$

$= \left[x\sin^{-1} x\right]_0^1 + \dfrac{1}{2}\int_0^1 \dfrac{(-2x)}{\sqrt{1-x^2}} dx$

Put, $1 - x^2 = t \Rightarrow -2xdx = dt$

When $x = 0, t = 1$ and when $x = 1, t = 0$

$I = \left[x\sin^{-1} x\right]_0^1 + \dfrac{1}{2}\int_0^1 \dfrac{dt}{\sqrt{t}}$

$= \left[x\sin^{-1} x\right]_0^1 + \dfrac{1}{2}\left[2\sqrt{t}\right]_1^0$

$= \sin^{-1}(1) + \left[-\sqrt{1}\right]$

$= \dfrac{\pi}{2} - 1$

Hence proved.

Question 40:

Evaluate $\int_0^1 e^{2-3x}dx$ as a limit of a sum.

Solution:

Let $I = \int_0^1 e^{2-3x}dx$

We know that,

$$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+...+f(a+(n-1)h)\right]$$

Where, $h = \dfrac{b-a}{n}$

Here, $a = 0, b = 1$ and $f(x) = e^{2-3x}$

$$\Rightarrow h = \frac{1-0}{n} = \frac{1}{n}$$

$$\therefore \int_0^1 e^{2-3x}dx = (1-0)\lim_{n\to\infty}\frac{1}{n}\left[f(0)+f(0+h)+...+f(0+(n-1)h)\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[e^2 + e^{2-3x} +...+ e^{2-3(n-1)h}\right] = \lim_{n\to\infty}\frac{1}{n}\left[e^2\left\{1+ e^{-3h} + e^{-6h} + e^{-9h} +...+ e^{-3(n-1)h}\right\}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[e^2\left\{\frac{1-\left(e^{-3h}\right)^n}{1-\left(e^{-3h}\right)}\right\}\right] = \lim_{n\to\infty}\frac{1}{n}\left[e^2\left\{\frac{1-e^{-\frac{3}{n}n}}{1-e^{-\frac{3}{n}}}\right\}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[\frac{e^2\left(1-e^{-3}\right)}{1-e^{-\frac{3}{n}}}\right] = e^2\left(e^{-3}-1\right)\lim_{n\to\infty}\frac{1}{n}\left[\frac{1}{e^{-\frac{3}{n}}-1}\right]$$

$$= e^2\left(e^{-3}-1\right)\lim_{n\to\infty}\left(-\frac{1}{3}\right)\left[\frac{-\dfrac{3}{n}}{e^{-\frac{3}{n}}-1}\right] = \frac{e^2\left(e^{-3}-1\right)}{3}\lim_{n\to\infty}\left[\frac{\dfrac{-3}{n}}{e^{-\frac{3}{n}}-1}\right]$$

$$= \frac{-e^2\left(e^{-3}-1\right)}{3}(1)$$

$$= \frac{-e^{-1}+e^2}{3}$$

$$= \frac{1}{3}\left(e^2 - \frac{1}{e}\right)$$

Question 41:

$\int\dfrac{dx}{e^x+e^{-x}}$ is equal to

A. $\tan^{-1}\left(e^x\right)+C$

B. $\tan^{-1}\left(e^{-x}\right)+C$

C. $\log\left(e^x-e^{-x}\right)+C$

D. $\log\left(e^x+e^{-x}\right)+C$

Solution:

Consider, $I=\int\dfrac{dx}{e^x+e^{-x}}dx=\int\dfrac{e^x}{e^{2x}+1}dx$

Put, $e^x=t\Rightarrow e^xdx=dt$

$\therefore I=\int\dfrac{dt}{1+t^2}$

$=\tan^{-1}t+C$

$=\tan^{-1}\left(e^x\right)+C$

Thus, the correct option is A.

Question 42:

$\int\dfrac{\cos 2x}{\left(\sin x+\cos x\right)^2}dx$ is

A. $\dfrac{-1}{\sin x+\cos x}+C$

B. $\log\left|\sin x+\cos x\right|+C$

C. $\log\left|\sin x-\cos x\right|+C$

D. $\dfrac{1}{\left(\sin x+\cos x\right)^2}+C$

Equals to

Solution:

Consider, $I=\int\dfrac{\cos 2x}{\left(\sin x+\cos x\right)^2}dx\Rightarrow I=\int\dfrac{\cos^2 x-\sin^2 x}{\left(\sin x+\cos x\right)}dx$

$=\int\dfrac{\left(\cos x+\sin x\right)\left(\cos x-\sin x\right)}{\left(\sin x+\cos x\right)^2}dx=\int\dfrac{\cos x-\sin x}{\cos x+\sin x}dx$

Let $\cos x+\sin x=t\Rightarrow\left(\cos x-\sin x\right)dx=dt$

$$\therefore I = \int \frac{dt}{t}$$

$$= \log|t| + C$$

$$= \log|\cos x + \sin x| + C$$

Thus, the correct option is B.

Question 43:

If $f(a+b-x) = f(x)$, then $\int_a^b xf(x)\,dx$ is equal to

A. $\dfrac{a+b}{2}\displaystyle\int_a^b f(b-x)\,dx$

B. $\dfrac{a+b}{2}\displaystyle\int_a^b f(b+x)\,dx$

C. $\dfrac{b-a}{2}\displaystyle\int_a^b f(x)\,dx$

D. $\dfrac{a+b}{2}\displaystyle\int_a^b f(x)\,dx$

Solution:

Consider, $I = \displaystyle\int_a^b xf(x)\,dx$ ...(1)

$I = \displaystyle\int_a^b (a+b-x)f(a+b-x)\,dx$ $\qquad \left(\displaystyle\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx \right)$

$\Rightarrow I = \displaystyle\int_a^b (a+b-x)f(x)\,dx$

$\Rightarrow I = (a+b)\displaystyle\int_a^b f(x)\,dx - I$......(Using equation ( ))

$\Rightarrow I + I = (a+b)\displaystyle\int_a^b f(x)\,dx$

$\Rightarrow 2I = (a+b)\displaystyle\int_a^b f(x)\,dx$

$\Rightarrow I = \left(\dfrac{a+b}{2}\right)\displaystyle\int_a^b f(x)\,dx$

Thus, the correct option is D.

Question 44:

The value of $\displaystyle\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx$ is

A. 1

B. 0

C. -1

D. $\dfrac{\pi}{4}$

Solution:

Consider, $I = \int_0^1 \tan^{-1}\left(\dfrac{2x-1}{1+x-x^2}\right) dx$

$\Rightarrow I = \int_0^1 \tan^{-1}\left(\dfrac{x-(1-x)}{1+x(1-x)}\right) dx$

$\Rightarrow I = \int_0^1 \left[\tan^{-1} x - \tan^{-1}(1-x)\right] dx \quad ...(1)$

$\Rightarrow I = \int_0^1 \left[\tan^{-1}(1-x) - \tan^{-1}(1-1+x)\right] dx$

$\Rightarrow I = \int_0^1 \left[\tan^{-1}(1-x) - \tan^{-1} x\right] dx$

$\Rightarrow I = \int_0^1 \left[\tan^{-1}(1-x) - \tan^{-1}(x)\right] dx \qquad ...(2)$

Adding (1) and (2), we get

$\Rightarrow 2I = \int_0^1 \left(\tan^{-1} x - \tan^{-1}(1-x) - \tan^{-1}(1-x) - \tan^{-1} x\right) dx$

$\Rightarrow 2I = 0$

$\Rightarrow I = 0$

Thus, the correct option is B.