Chapter 6 Applications of Derivatives

EXERCISE 6.1

Question 1:

Find the rate of change of the area of a circle with respect to its radius r when

(a) $r = 3cm$

(b) $r = 4cm$

Solution:

We know that the area of a circle, $A = \pi r^2$

Therefore, the rate of change of the area with respect to its radius is given by

$$\frac{dA}{dr} = \frac{d}{dr}\left(\pi r^2\right)$$
$$= 2\pi r$$

(a) When $r = 3cm$

Then,

$$\frac{dA}{dr} = 2\pi(3)$$
$$= 6\pi$$

Thus, the area is changing at the rate of 6π.

(b) When $r = 4cm$

Then,

$$\frac{dA}{dr} = 2\pi(4)$$
$$= 8\pi$$

Thus, the area is changing at the rate of 8π.

Question 2:

The volume of a cube is increasing at the rate of $8cm^3/s$. How fast is the surface area increasing when the length of its edge is $12cm$?

Solution:

Let the side length, volume and surface area respectively be equal to x, V and S.

Hence, $V = x^3$ and $S = 6x^2$

We have,

$$\frac{dV}{dt} = 8cm^3 / s$$

Therefore,

$$\frac{dV}{dt} = \frac{d}{dt}\left(x^3\right)$$

$$8 = \frac{d}{dx}\left(x^3\right)\frac{dx}{dt}$$

$$8 = 3x^2 \frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{8}{3x^2} \qquad \ldots(1)$$

Now,

$$\frac{dS}{dt} = \frac{d}{dt}\left(6x^2\right)$$

$$= \frac{d}{dx}\left(6x^2\right)\frac{dx}{dt}$$

$$= 12x\frac{dx}{dt}$$

$$= 12x\left(\frac{8}{3x^2}\right) \qquad \left[\text{from } (1)\right]$$

$$= \frac{32}{x}$$

So, when $x = 12cm$

Then,

$$\frac{dS}{dt} = \frac{32}{12}cm^2 / s$$

$$= \frac{8}{3}cm^2 / s$$

Question 3:

The radius of a circle is increasing uniformly at the rate of $3cm/s$. Find the rate at which the area of the circle is increasing when the radius is $10cm/s$.

Solution:

We know that $A = \pi r^2$

Now,

$$\frac{dA}{dt} = \frac{d}{dr}\left(\pi r^2\right)$$
$$= 2\pi r \frac{dr}{dt}$$

We have,

$$\frac{dr}{dt} = 3cm/s$$

Hence,

$$\frac{dA}{dt} = 2\pi r(3)$$
$$= 6\pi r$$

So, when $r = 10cm$

Then,

$$\frac{dA}{dt} = 6\pi(10)$$
$$= 60\pi cm^2/s$$

Question 4:

An edge of a variable cube is increasing at the rate of $3cm/s$. How fast is the volume of the cube increasing when the edge is $10cm$ long?

Solution:

Let the length and the volume of the cube respectively be x and V .

Hence, $V = x^3$

Now,

$$\frac{dV}{dt} = \frac{d}{dt}\left(x^3\right)$$
$$= \frac{d}{dx}\left(x^3\right)\frac{dx}{dt}$$
$$= 3x^2\frac{dx}{dt}$$

We have,

$$\frac{dx}{dt} = 3cm/s$$

Hence,

$$\frac{dV}{dt} = 3x^2(3)$$
$$= 9x^2$$

So, when $x = 10cm$

Then,

$$\frac{dV}{dt} = 9(10)^2$$
$$= 900cm^3/s$$

Question 5:

A stone is dropped into a quiet lake and waves move in circles at the speed of $5cm/s$. At the instant when the radius of the circular wave is $8cm$, how fast is the encoding area is increasing?

Solution:

We know that $A = \pi r^2$

Now,

$$\frac{dA}{dt} = \frac{d}{dt}(\pi r^2)$$
$$= 2\pi r \frac{dr}{dt}$$

We have,

$$\frac{dr}{dt} = 5cm/s$$

Hence,

$$\frac{dA}{dt} = 2\pi r(5)$$
$$= 10\pi r$$

So, when $r = 8cm$

Then,

$$\frac{dA}{dt} = 10\pi(8)$$
$$= 80\pi cm^2/s$$

Question 6:

The radius of a circle is increasing at the rate of $0.7 cm/s$. What is the rate of increase of its circumference?

Solution:

We know that $C = 2\pi r$

Now,

$$\frac{dC}{dt} = \frac{d}{dt}(2\pi r)$$
$$= \frac{d}{dr}(2\pi r)\frac{dr}{dt}$$
$$= 2\pi \frac{dr}{dt}$$

We have,

$$\frac{dr}{dt} = 0.7\pi cm/s$$

Hence,

$$\frac{dC}{dt} = 2\pi(0.7)$$
$$= 1.4\pi cm/s$$

Question 7:

The length x of a rectangle is decreasing at the rate of $5cm/minute$ and the width y is increasing at the rate of $4cm/minute$. When $8cm$ and $y = 6cm$, find the rate of change of **(a)** the perimeter and **(b)** the area of the rectangle.

Solution:

It is given that $\frac{dx}{dt} = -5cm/minute$, $\frac{dx}{dt} = 4cm/minute$, $x = 8cm$ and $y = 6cm$

(a) The perimeter of a rectangle is given by $P = 2(x+y)$
Therefore,

$$\frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right)$$
$$= 2(-5+4)$$
$$= -2cm/minute$$

$$\frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right)$$
$$= 2(-5+4)$$
$$= -2cm / minute$$

(b) The area of a rectangle is given by $A = xy$
Therefore,

$$\frac{dA}{dt} = \frac{dx}{dt}y + x\frac{dy}{dt}$$
$$= -5y + 4x$$
$$\frac{dA}{dt} = \frac{dx}{dt}y + x\frac{dy}{dt}$$
$$= -5y + 4x$$

When $x = 8cm$ and $y = 6cm$
Then,

$$\frac{dA}{dt} = (-5\times 6 + 4\times 8)cm^2 / minute$$
$$= 2cm^2 / minute$$

Question 8:

A balloon, which always remains spherical on inflation, is being inflated by pumping in 900 cubic centimeters of gas per second. Find the rate at which the radius of the balloon increases when the radius is $15cm$.

Solution:

We know that $V = \frac{4}{3}\pi r^3$
Hence,

$$\frac{dV}{dt} = \frac{dV}{dt}\left(\frac{4}{3}\pi r^3\right)$$
$$= \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right)\frac{dr}{dt}$$
$$= 4\pi r^2\frac{dr}{dt}$$

We have,

$$\frac{dV}{dt} = 900cm^2 / s$$

Therefore,

$$4\pi r^2 \frac{dr}{dt} = 900$$

$$\frac{dr}{dt} = \frac{900}{4\pi r^2}$$

$$= \frac{225}{\pi r^2}$$

When radius, $r = 15cm$

Then,

$$\frac{dr}{dt} = \frac{225}{\pi (15)^2}$$

$$= \frac{1}{\pi} cm / s$$

Question 9:

A balloon, which always remains spherical has a variable radius. Find the rate at which its volume is increasing with the radius when the latter is $10cm$.

Solution:

We know that $V = \frac{4}{3}\pi r^2$

Therefore,

$$\frac{dV}{dt} = \frac{d}{dr}\left(\frac{4}{3}\pi r^2\right)$$

$$= \frac{4}{3}\pi \left(3r^2\right)$$

$$= 4\pi r^2$$

When radius, $r = 10cm$

Then,

$$\frac{dV}{dt} = 4\pi (10)^2$$

$$= 400\pi$$

Thus, the volume of the balloon is increasing at the rate of $400\pi cm^3 / s$.

Question 10:

A ladder is $5m$ long is leaning against the wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of $2cm/s$. How fast is its height on the wall decreasing when the foot of the ladder is $4m$ away from the wall?

Solution:

Let the height of the wall at which the ladder is touching it be ym and the distance of its foot from the wall on the ground be xm.

Hence,

$$x^2+y^2=5^2$$
$$\Rightarrow y^2=25-x^2$$
$$\Rightarrow y=\sqrt{25-x^2}$$

Therefore,

$$\frac{dy}{dt}=\frac{d}{dt}\left(\sqrt{25-x^2}\right)$$
$$=\frac{d}{dx}\left(\sqrt{25-x^2}\right)\frac{dx}{dt}$$
$$=\frac{-x}{\left(\sqrt{25-x^2}\right)}\frac{dx}{dt}$$

We have,

$$\frac{dx}{dt}=2cm/s$$

Thus,

$$\frac{dy}{dt}=\frac{-2x}{\sqrt{25-x^2}}$$

When $x=4cm$

Then,

$$\frac{dy}{dt}=\frac{-2\times 4}{\sqrt{25-16}}$$
$$=-\frac{8}{3}cm/s$$

Question 11:

A particle is moving along the curve $6y=x^3+2$. Find the points on the curve at which the y-coordinate is changing 8 times as fast as the x-coordinate.

Solution:

The equation of the curve is $6y = x^3 + 2$

Differentiating with respect to time, we have

$$\Rightarrow 6\frac{dy}{dt} = 3x^2\frac{dx}{dt}$$
$$\Rightarrow 2\frac{dy}{dt} = x^2\frac{dx}{dt}$$

According to the question, $\left(\frac{dy}{dt} = 8\frac{dx}{dt}\right)$
Hence,

$$\Rightarrow 2\left(8\frac{dx}{dt}\right) = x^2\frac{dx}{dt}$$
$$\Rightarrow 16\frac{dx}{dt} = x^2\frac{dx}{dt}$$
$$\Rightarrow \left(x^2 - 16\right)\frac{dx}{dt} = 0$$
$$\Rightarrow x^2 = 16$$
$$\Rightarrow x = \pm 4$$

When $x = 4$
Then,

$$y = \frac{4^3 + 2}{6}$$
$$= \frac{66}{6}$$
$$= 11$$

When $x = -4$
Then,

$$y = \frac{\left(-4^3\right) + 2}{6}$$
$$= -\frac{62}{6}$$
$$= -\frac{31}{3}$$

Thus, the points on the curve are $(4, 11)$ and $\left(-4, \frac{-31}{3}\right)$.

Question 12:

The radius of an air bubble is increasing at the rate of $\frac{1}{2}cm/s$. At which rate is the volume of the bubble increasing when the radius is $1cm$?

Solution:

Assuming that the air bubble is a sphere, $V=\frac{4}{3}\pi r^3$
Therefore,

$$\frac{dV}{dt}=\frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)\frac{dr}{dt}$$
$$=4\pi r^2\frac{dr}{dt}$$

We have,

$$\frac{dr}{dt}=\frac{1}{2}cm/s$$

When $r=1cm$

Then,

$$\frac{dV}{dt}=4\pi(1)^2\left(\frac{1}{2}\right)$$
$$=2\pi cm^3/s$$

Question 13:

A balloon, which always remains spherical, has a variable diameter $\frac{3}{2}(2x+1)$. Find the rate of change of its volume with respect to x.

Solution:

We know that $V=\frac{4}{3}\pi r^3$

It is given that diameter, $d=\frac{3}{2}(2x+1)$

Hence, $r=\frac{3}{4}(2x+1)$

Therefore,

$$V=\frac{4}{3}\pi\left(\frac{3}{4}\right)^3(2x+1)^3$$
$$=\frac{9}{16}\pi(2x+1)^3$$

Thus,

$$\frac{dV}{dx}=\frac{9}{16}\pi\frac{d}{dx}(2x+1)^3$$
$$=\frac{27}{8}\pi(2x+1)^3$$

Question 14:

Sand is pouring from a pipe at the rate of $12cm^3/s$. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when height is $4cm$?

Solution:

We know that $V=\frac{1}{3}\pi r^2h$

It is given that, $h=\frac{1}{6}r$

Hence, $r=6h$

Therefore,

$$V=\frac{1}{3}\pi(6h)^2h$$
$$=12\pi h^3$$

Thus,

$$\frac{dV}{dt}=12\pi\frac{d}{dt}\left(h^3\right)\frac{dh}{dt}$$
$$=12\pi\left(3h^2\right)\frac{dh}{dt}$$
$$=36\pi h^2\frac{dh}{dt}$$

We have,

$$\frac{dV}{dt}=12cm^2/s$$

When $h = 4cm$

Then,

$$12 = 36\pi(4)^2 \frac{dh}{dt}$$
$$\frac{dh}{dt} = \frac{12}{36\pi(16)}$$
$$= \frac{1}{48\pi} cm/s$$

Question 15:

The total cost $C(x)$ in Rupees associated with the production of x units of an item is given by $C(x) = 0.007x^3 - 0.003x^2 + 15x + 4000$. Find the marginal cost when 17 units are produced.

Solution:

Marginal cost (MC) is the rate of change of the total cost with respect to the output.
Therefore,

$$MC = \frac{dC}{dx} = 0.007(3x^2) - 0.003(2x) + 15$$
$$= 0.021x^2 - 0.006x + 15$$

When $x = 17$
Then,

$$MC = 0.021(17)^2 - 0.006(17) + 15$$
$$= 0.021(289) - 0.006(17) + 15$$
$$= 6.069 - 0.102 + 15$$
$$= 20.967$$

So, when 17 units are produced, the marginal cost is ₹ 20.967.

Question 16:

The total revenue in Rupees received from the sale of x units of a product given by $R(x) = 13x^2 + 26x + 15$. Find the marginal revenue when $x = 7$.

Solution:

Marginal revenue (MR) is the rate of change of the total revenue with respect to the number of units sold.
Therefore,

$$MR = \frac{dR}{dx} = 13(2x) + 26$$
$$= 26x + 26$$

When, $x = 7$
Then,

$$MR = 26(7) + 26$$
$$= 182 + 26$$
$$= 208$$

Thus, the marginal revenue is ₹ 208.

Question 17:

The rate of change of the area of a circle with respect to its radius r at $r = 6cm$ is

(A) 10π (B) 12π (C) 8π (D) 11π

Solution:

We know that $A = \pi r^2$

Therefore,

$$\frac{dA}{dr} = \frac{d}{dr}(\pi r^2)$$
$$= 2\pi r$$

When $r = 6cm$

Then,

$$\frac{dA}{dr} = 2\pi \times 6$$
$$= 12\pi cm^2 / s$$

Thus, the rate of change of the area of the circle is $12\pi cm^2 / s$.

Hence, the correct option is **B**.

Question 18:

The total revenue is Rupees received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$. The marginal revenue, when $x = 15$ is

(A) 116 (B) 96 (C) 90 (D) 126

Solution:

Marginal revenue (MR) is the rate of change of the total revenue with respect to the number of units sold.

Therefore,

$$MR = \frac{dR}{dx} = 3(2x) + 36$$
$$= 6x + 36$$

When, $x = 15$

Then,

$$MR = 6(15) + 36$$
$$= 90 + 36$$
$$= 126$$

Thus, the marginal revenue is ₹ 126.

Hence, the correct option is **D**.

EXERCISE 6.2

Question 1:

Show that the function given by $f(x)=3x+17$ is strictly increasing on **R**.

Solution:

Let x_1 and x_2 be any two numbers in **R**.
Then,

$$x_1 < x_2 \Rightarrow 3x_1 + 17 < 3x_2 + 17 = f(x_1) < f(x_2)$$

Thus, f is strictly increasing on **R**.

Question 2:

Show that the function given by $f(x)=e^{2x}$ is strictly increasing on **R**.

Solution:

Let x_1 and x_2 be any two numbers in **R**.
Then,

$$x_1 < x_2 \Rightarrow 2x_1 < 2x_2 \Rightarrow e^{2x_1} < e^{2x_2} = f(x_1) < f(x_2)$$

Thus, f is strictly increasing on **R**.

Question 3:

Show that the function given by $f(x)=\sin x$ is

(a) Strictly increasing in $\left(0,\frac{\pi}{2}\right)$

(b) Strictly decreasing in $\left(\frac{\pi}{2},\pi\right)$

(c) Neither increasing nor decreasing in $(0,\pi)$

Solution:

It is given that $f(x)=\sin x$

Hence, $f'(x)=\cos x$

(a) Here, $x \in \left(0,\frac{\pi}{2}\right)$

$$\Rightarrow \cos x > 0$$
$$\Rightarrow f'(x) > 0$$

Thus, f is strictly increasing in $\left(0, \frac{\pi}{2}\right)$.

(b) Here, $x \in \left(\frac{\pi}{2}, \pi\right)$

$$\Rightarrow \cos x < 0$$
$$\Rightarrow f'(x) < 0$$

Thus, f is strictly decreasing in $\left(\frac{\pi}{2}, \pi\right)$.

(c) Here, $x \in (0, \pi)$

The results obtained in **(a)** and **(b)** are sufficient to state that f is neither increasing nor decreasing in $(0, \pi)$.

Question 4:

Find the intervals in which the function f given by $f(x) = 2x^2 - 3x$ is
(a) Strictly increasing **(b)** Strictly decreasing

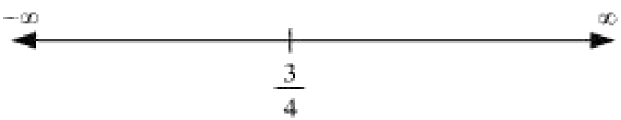
Solution:

The given function is $f(x) = 2x^2 - 3x$
Hence,

$$f'(x) = 4x - 3$$

Therefore,

$$f'(x) = 0$$
$$\Rightarrow x = \frac{3}{4}$$



In $\left(-\infty, \frac{3}{4}\right)$, $f'(x) = 4x - 3 < 0$

Hence, f is strictly decreasing in $\left(-\infty, \frac{3}{4}\right)$.

In $\left(\frac{3}{4}, \infty\right)$, $f'(x) = 4x - 3 > 0$

Hence, f is strictly increasing in $\left(\frac{3}{4}, \infty\right)$.

Question 5:

Find the intervals in which the function f given $f(x) = 2x^3 - 3x^2 - 36x + 7$ is

(a) Strictly increasing **(b)** Strictly decreasing

Solution:

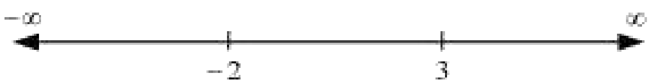
The given function is $f(x) = 2x^3 - 3x^2 - 36x + 7$

Hence,

$$\begin{aligned} f'(x) &= 6x^2 - 6x - 36 \\ &= 6(x^2 - x - 6) \\ &= 6(x+2)(x-3) \end{aligned}$$

Therefore,

$$\begin{aligned} &f'(x) = 0 \\ &\Rightarrow x = -2, 3 \end{aligned}$$



In $(-\infty, -2)$ and $(3, \infty), f'(x) > 0$

In $(-2, 3), f'(x) < 0$

Hence, f is strictly increasing in $(-\infty, -2), (3, \infty)$ and strictly decreasing in $(-2, 3)$.

Question 6:

Find the intervals in which the following functions are strictly increasing or decreasing.

(a) $x^2 + 2x - 5$ **(b)** $10 - 6x - 2x^2$

(c) $-2x^3 - 9x^2 - 12x + 1$ **(d)** $6 - 9x - 9x^2$

(e) $(x+1)^3 (x-3)^3$

Solution:

(a) $f(x) = x^2 + 2x - 5$

Hence,

$$f'(x) = 2x + 2$$

Therefore,

$$\Rightarrow f'(x) = 0$$
$$\Rightarrow x = -1$$

$x = -1$ divides the number line into intervals $(-\infty, 1)$ and $(-1, \infty)$

In $(-\infty, 1), f'(x) = 2x + 2 < 0$

Thus, f is strictly decreasing in $(-\infty, 1)$

In $(-1, \infty), f'(x) = 2x + 2 > 0$

Thus, f is strictly increasing in $(-1, \infty)$

(b) $f(x) = 10 - 6x - 2x^2$

Hence,

$$f'(x) = -6 - 4x$$

Therefore,

$$\Rightarrow f'(x) = 0$$
$$\Rightarrow x = -\frac{3}{2}$$

$x = -\frac{3}{2}$, divides the number line into two intervals $\left(-\infty, -\frac{3}{2}\right)$ and $\left(-\frac{3}{2}, \infty\right)$

In $\left(-\infty, -\frac{3}{2}\right), f'(x) = -6 - 4x < 0$

Hence, f is strictly increasing for $x < -\frac{3}{2}$

In $\left(-\frac{3}{2}, \infty\right), f'(x) = -6 - 4x > 0$

Hence, f is strictly increasing for $x > -\frac{3}{2}$

(c) $f(x) = -2x^3 - 9x^2 - 12x + 1$

Hence,

$$f'(x) = -6x^2 - 18x - 12$$
$$= -6(x^2 + 3x + 2)$$
$$= -6(x+1)(x+2)$$

Therefore,

$$\Rightarrow f'(x) = 0$$
$$\Rightarrow x = -1, 2$$

$x = -1$ and $x = -2$ divide the number line into intervals $(-\infty, -2)$, $(-2, -1)$ and $(-1, \infty)$.

In $(-\infty, -2)$ and $(-1, \infty)$, $f'(x) = -6(x+1)(x+2) < 0$

Hence, f is strictly decreasing for $x < -2$ and $x > -1$

In $(-2, -1), f'(x) = -6(x+1)(x+2) > 0$

Hence, f is strictly increasing in $-2 < x < -1$

(d) $f(x) = 6 - 9x - x^2$

Hence,

$$f'(x) = -9 - 2x$$

Therefore,

$$\Rightarrow f'(x) = 0$$
$$\Rightarrow x = -\frac{9}{2}$$

In $\left(-\frac{9}{2}, \infty\right), f'(x) < 0$

Hence, f is strictly decreasing for $x > -\frac{9}{2}$

In $\left(-\infty, -\frac{9}{2}\right), f'(x) > 0$

Hence, f is strictly decreasing in $x > -\frac{9}{2}$

(e) $f(x) = (x+1)^3 (x-3)^3$

Hence,

$$\begin{aligned}f'(x) &= 3(x+1)^2(x-3)^3 + 3(x-3)^2(x+1)^3 \\ &= 3(x+1)^2(x-3)^2[x-3+x+1] \\ &= 3(x+1)^2(x-3)^2(2x-2) \\ &= 6(x+1)^2(x-3)^2(x-1)\end{aligned}$$

Therefore,

$$f'(x) = 0$$
$$\Rightarrow x = -1, 3, 1$$

$x = -1, 3, 1$ divides the number line into four intervals $(-\infty, -1), (-1, 1), (1, 3)$ and $(3, \infty)$

In $(-\infty, -1)$ and $(-1, 1)$, $f'(x) = 6(x+1)^2(x-3)^2(x-1) < 0$

Hence, f is strictly decreasing in $(-\infty, -1)$ and $(-1, 1)$

In $(1, 3)$ and $(3, \infty)$, $f'(x) = 6(x+1)^2(x-3)^2(x-1) > 0$

Hence, f is strictly increasing in $(1, 3)$ and $(3, \infty)$

Question 7:

Show that $y = \log(1+x) - \frac{2x}{2+x}, x > -1$, is an increasing function of x throughout its domain.

Solution:

It is given that $y = \log(1+x) - \frac{2x}{2+x}$

Therefore,

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{1+x} - \frac{(2+x)(2) - 2x(1)}{(2+x)^2} \\ &= \frac{1}{1+x} - \frac{4}{(2+x)^2} \\ &= \frac{x^2}{(1+x)(2+x)^2}\end{aligned}$$

Now, $\frac{dy}{dx} = 0$

Hence,

$$\Rightarrow \frac{x^2}{(2+x)^2} = 0$$
$$\Rightarrow x^2 = 0$$
$$\Rightarrow x = 0$$

Since, $x > -1, x = 0$ divides domain $(-1, \infty)$ in two intervals $-1 < x < 0$ and $x > 0$

When, $-1 < x < 0$
Then,

$$x < 0 \Rightarrow x^2 > 0$$
$$x > -1 \Rightarrow (2+x) > 0$$
$$\Rightarrow (2+x)^2 > 0$$

Hence,

$$y = \frac{x^2}{(2+x)^2} > 0$$

When, $x > 0$
Then,

$$x > 0 \Rightarrow x^2 > 0$$
$$\Rightarrow (2+x)^2 > 0$$

Hence,

$$y = \frac{x^2}{(2+x)^2} > 0$$

Thus, f is increasing throughout the domain.

Question 8:

Find the values of x for which $y=\left[x(x-2)\right]^2$ is an increasing function.

Solution:

We have,

$$y=\left[x(x-2)\right]^2$$
$$=\left[x^2-2x\right]^2$$

Therefore,

$$\frac{dy}{dx}=\frac{d}{dx}\left[x^2-2x\right]^2$$
$$=2\left(x^2-2x\right)(2x-2)$$
$$=4x(x-2)(x-1)$$

Now, $\frac{dy}{dx}=0$

Hence,

$$\Rightarrow 4x(x-2)(x-1)$$
$$\Rightarrow x=0, x=2, x=1$$

$x=0$, $x=1$ and $x=2$ divide the number line intervals $(-\infty,0),(0,1),(1,2)$ and $(2,\infty)$

In $(-\infty,0)$ and $(1,2)$, $\frac{dy}{dx}<0$

Hence, y is strictly decreasing in intervals $(-\infty,0)$ and $(1,2)$

In $(0,1)$ and $(2,\infty)$, $\frac{dy}{dx}>0$

Hence, y is strictly increasing in intervals $(0,1)$ and $(2,\infty)$

Question 9:

Prove that $y=\frac{4\sin\theta}{(2+\cos\theta)}-\theta$ is an increment function of θ in $\left[0,\frac{\pi}{2}\right]$.

Solution:

We have, $y = \frac{4\sin\theta}{(2+\cos\theta)} - \theta$

Therefore,

$$\frac{dy}{d\theta} = \frac{(2+\cos\theta)(4\cos\theta) - 4\sin\theta(-\sin\theta)}{(2+\cos\theta)^2} - 1$$
$$= \frac{8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}{(2+\cos\theta)^2} - 1$$
$$= \frac{8\cos\theta + 4}{(2+\cos\theta)^2} - 1$$

Now, $\frac{dy}{d\theta} = 0$

Hence,

$$\Rightarrow \frac{8\cos\theta}{(2+\cos\theta)^2} = 1$$
$$\Rightarrow 8\cos\theta + 4 = 4 + \cos^2\theta + 4\cos\theta$$
$$\Rightarrow \cos^2\theta - 4\cos\theta = 0$$
$$\Rightarrow \cos\theta(\cos\theta - 4) = 0$$
$$\Rightarrow \cos\theta = 0 \text{ or } \cos\theta = 4$$

Since, $\cos\theta \neq 4$

Therefore,

$$\cos\theta = 0$$
$$\Rightarrow \theta = \frac{\pi}{2}$$

Now,

$$\frac{dy}{d\theta} = \frac{8\cos\theta + 4 - (4 + \cos^2\theta + 4\cos\theta)}{(2+\cos\theta)^2}$$
$$= \frac{4\cos\theta - \cos^2\theta}{(2+\cos\theta)^2}$$
$$= \frac{\cos(4-\cos\theta)}{(2+\cos\theta)^2}$$

In interval $\left[0, \frac{\pi}{2}\right]$, we have $\cos\theta > 0$
Also,

$$4 > \cos\theta$$
$$\Rightarrow 4 - \cos\theta > 0$$

Hence, $\cos\theta(4-\cos\theta) > 0$ and also $(2+\cos\theta)^2 > 0$

Therefore,

$$\frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2} > 0$$

Hence, $\frac{dy}{d\theta} > 0$

So, y is strictly increasing in $\left(0, \frac{\pi}{2}\right)$ and the given function is continuous at $x = 0$ and $x = \frac{\pi}{2}$

Thus, y is increasing in interval $\left[0, \frac{\pi}{2}\right]$.

Question 10:

Prove that the logarithmic function is strictly increasing on $(0, \infty)$.

Solution:

The given function is $f(x) = \log x$

Therefore, $f'(x) = \frac{1}{x}$

For, $x > 0, f'(x) = \frac{1}{x} > 0$

Thus, the logarithmic function is strictly increasing in interval $(0, \infty)$.

Question 11:

Prove that the function f given by $f(x) = x^2 - x + 1$ is neither strictly increasing nor strictly decreasing on $(-1,1)$.

Solution:

The given function is $f(x)=x^2-x+1$

Therefore,

$$f'(x)=2x-1$$

Now,

$$f'(x)=0$$
$$\Rightarrow x=\frac{1}{2}$$

$x=\frac{1}{2}$ divides the interval v into $\left(-1,\frac{1}{2}\right)$ and $\left(\frac{1}{2},1\right)$

In interval $\left(\frac{1}{2},1\right), f'(x)=2x-1>0$

Hence, f is strictly decreasing in $\left(-1,\frac{1}{2}\right)$

In interval $\left(\frac{1}{2},1\right), f'(x)=2x-1>0$

Hence, f is strictly increasing in $\left(\frac{1}{2},1\right)$

Thus, f is strictly increasing nor strictly decreasing in interval $(-1,1)$

Question 12:

Which of the following principles are strictly decreasing on $\left(0,\frac{\pi}{2}\right)$?

(A) $\cos x$ **(B)** $\cos 2x$ **(C)** $\cos 3x$ **(D)** $\tan x$

Solution:

(A) Let $f_1(x)=\cos x$

Therefore, $f_1'(x)=-\sin x$

In interval $\left(0,\frac{\pi}{2}\right), f_1'(x)=-\sin x<0$

Thus, $\cos x$ is strictly decreasing in $\left(0,\frac{\pi}{2}\right)$.

(B) Let $f_2(x)=\cos 2x$

Therefore, $f_2'(x)=-2\sin 2x$

Now,

$$\begin{aligned}&\Rightarrow 0<x<\frac{\pi}{2}\\&\Rightarrow 0<2x<\pi\\&\Rightarrow \sin 2x>0\\&\Rightarrow -2\sin 2x<0\end{aligned}$$

Hence, $f_2'(x)=-2\sin 2x<0$ in $\left(0,\frac{\pi}{2}\right)$

Thus, $\cos 2x$ is strictly decreasing in $\left(0,\frac{\pi}{2}\right)$

(C) Let $f_3(x)=\cos 3x$

Therefore, $f_3'(x)=-3\sin 3x$

Now,

$$\begin{aligned}&f_3'(x)=0\\&\Rightarrow \sin 3x=0\\&\Rightarrow 3x=\pi \qquad \left[\because x\in\left(0,\frac{\pi}{2}\right)\right]\\&\Rightarrow x=\frac{\pi}{3}\end{aligned}$$

The point $x=\frac{\pi}{3}$, divides $\left(0,\frac{\pi}{2}\right)$ into $\left(0,\frac{\pi}{3}\right)$ and $\left(\frac{\pi}{3},\frac{\pi}{2}\right)$

In interval $\left(0,\frac{\pi}{3}\right), f_3(x)=-3\sin 3x<0 \quad \left[\begin{matrix}0<x<\frac{\pi}{3}\\0<3x<\pi\end{matrix}\right]$

Hence, f_3 is strictly decreasing in $\left(0,\frac{\pi}{3}\right)$

In interval $\left(\frac{\pi}{3},\frac{\pi}{2}\right), f_3(x)=-3\sin 3x>0 \quad \left[\frac{\pi}{3}<x<\frac{\pi}{2}<\pi<3x<\frac{3\pi}{2}\right]$

Hence, f_3 is strictly increasing in $\left(\frac{\pi}{3},\frac{\pi}{2}\right)$

Thus, $\cos 3x$ is neither increasing nor decreasing in interval $\left(0,\frac{\pi}{2}\right)$

(D) Let $f_4(x)=\tan x$

Therefore, $f_4'(x)=\sec^2 x$

In interval $\left(0,\frac{\pi}{2}\right), f_4'(x)=\sec^2 x>0$

Thus, $\tan x$ is strictly increasing in $\left(0,\frac{\pi}{2}\right)$

Thus, the correct options are **A** and **B**.

Question 13:

On which of the following intervals is the function f is given by $f(x)=x^{100}+\sin x-1$ is strictly decreasing?

(A) $(0,1)$ **(B)** $\left(\frac{\pi}{2},\pi\right)$ **(C)** $\left(0,\frac{\pi}{2}\right)$ **(D)** None of these

Solution:

We have,

$$f(x)=x^{100}+\sin x-1$$

Therefore,

$$f'(x)=100x^{99}+\cos x$$

In interval $(0,1), \cos x>0$ and $100x^{99}>0$

Hence, $f'(x)>0$

Thus, f is strictly increasing in $(0,1)$

In interval $\left(\frac{\pi}{2},\pi\right), \cos x < 0$ and $100x^{99} > 0$

Hence, $f'(x) > 0$

Thus, f is strictly increasing in interval $\left(\frac{\pi}{2},\pi\right)$

Now, in interval $\left(0,\frac{\pi}{2}\right), \cos x > 0$ and $100x^{99} > 0$

Hence, $f'(x) > 0$

Thus, f is strictly increasing in interval $\left(0,\frac{\pi}{2}\right)$

Hence, f is strictly decreasing in none of the intervals.

Thus, the correct option is **D**.

Question 14:

For what values of a the function f given $f(x) = x^2 + ax + 1$ is increasing on $[1,2]$?

Solution:

We have

$$f(x) = x^2 + ax + 1$$

Therefore,

$$f'(x) = 2x + a$$

Now, the function f is strictly increasing on $[1,2]$

Therefore,

$$\Rightarrow f'(x) > 0$$
$$\Rightarrow 2x + a > 0$$
$$\Rightarrow 2x > -a$$
$$\Rightarrow x > \frac{-a}{2}$$

Here, we have $1 \leq x \leq 2$

Thus,

$$\frac{-a}{2} > 1$$
$$a > -2$$

Question 15:

Let **I** be any interval disjoint from $[-1,1]$. Prove that the function f given by $f(x) = x + \frac{1}{x}$ is increasing on **I**.

Solution:

We have

$$f(x) = x + \frac{1}{x}$$

Therefore,

$$f'(x) = 1 - \frac{1}{x^2}$$

Now,

$$f'(x) = 0$$
$$\Rightarrow 1 - \frac{1}{x^2} = 0$$
$$\Rightarrow x^2 = 1$$
$$\Rightarrow x = \pm 1$$

The points $x = 1$ and $x = -1$ divide the real line intervals $(-\infty, 1), (-1, 1)$ and $(1, \infty)$

In interval $(-1, 1)$, $-1 < x < 1$

$$\Rightarrow x^2 < 1$$
$$\Rightarrow 1 < \frac{1}{x^2} \qquad x \neq 0$$
$$\Rightarrow 1 - \frac{1}{x^2} < 0 \qquad x \neq 0$$

Therefore, $f'(x) = 1 - \frac{1}{x^2} < 0$ on $(-1, 1) \sim \{0\}$

Hence, f is strictly decreasing on $(-1, 1) \sim \{0\}$

Now, in interval $(-\infty, -1)$ and $(1, \infty)$, $x < -1$ or $1 < x$

$$\Rightarrow x^2 > 1$$

$$\Rightarrow 1 > \frac{1}{x^2}$$

$$\Rightarrow 1 - \frac{1}{x^2} > 0$$

Therefore, $f'(x) = 1 - \frac{1}{x^2} > 0$ on $(-\infty, -1)$ and $(1, \infty)$

Hence, f is strictly increasing on $(-\infty, -1)$ and $(1, \infty)$

Thus, f is strictly increasing in **I** in $[-1, 1]$

Question 16:

Prove that the function f given by $f(x) = \log \sin x$ is increasing on $\left(0, \frac{\pi}{2}\right)$ and decreasing on $\left(\frac{\pi}{2}, \pi\right)$.

Solution:

We have

$$f(x) = \log \sin x$$

Therefore,

$$\begin{aligned} f'(x) &= \frac{1}{\sin x} \cos x \\ &= \cot x \end{aligned}$$

In interval $\left(0, \frac{\pi}{2}\right), f'(x) = \cot x > 0$

Hence, f is strictly increasing in $\left(0, \frac{\pi}{2}\right)$.

In interval $\left(\frac{\pi}{2}, \pi\right), f'(x) = \cot x < 0$

Hence, f is strictly decreasing in $\left(\frac{\pi}{2}, \pi\right)$.

Question 17:

Prove that the function f given by $f(x)=\log|\cos x|$ is decreasing on $\left(0,\frac{\pi}{2}\right)$ and increasing on $\left(\frac{3\pi}{2},2\pi\right)$.

Solution:

We have $f(x)=\log|\cos x|$

Therefore,

$$f'(x)=\frac{1}{\cos x}(-\sin x)$$
$$=-\tan x$$

In interval $\left(0,\frac{\pi}{2}\right)$, $\tan x>0 \Rightarrow -\tan x<0$

Hence, $f'(x)<0$

Thus, f is strictly decreasing on $\left(0,\frac{\pi}{2}\right)$.

In interval $\left(\frac{3\pi}{2},2\pi\right)$, $\tan x<0 \Rightarrow -\tan x>0$

Hence, $f'(x)>0$

Thus, f is strictly increasing on $\left(\frac{3\pi}{2},2\pi\right)$.

Question 18:

Prove that the function given by $f(x)=x^3-3x^2+3x-100$ is increasing in **R**.

Solution:

We have

$$f(x)=x^3-3x^2+3x-100$$

Therefore,

$$f'(x)=3x^2-6x+3$$
$$=3(x^2-2x+1)$$
$$=3(x-1)^2$$

For $x \in \mathbf{R}$, $(x-1)^2 \geq 0$

So, $f'(x)$ is always positive in **R**.

Thus, the function is increasing in **R**.

Question 19:

The interval in which $y = x^2 e^{-x}$ is increasing is

(A) $(-\infty, \infty)$ **(B)** $(-2, 0)$ **(C)** $(2, \infty)$ **(D)** $(0, 2)$

Solution:

We have $y = x^2 e^{-x}$

Therefore,

$$\frac{dy}{dx} = 2xe^{-x} - x^2 e^{-x}$$
$$= xe^{-x}(2-x)$$

Now, $\frac{dy}{dx} = 0$

Hence, $x = 0$ and $x = 2$

The points $x = 0$ and $x = 2$ divide the real line into three disjoint intervals i.e., $(-\infty, 0)$, $(0, 2)$ and $(2, \infty)$.

In intervals $(-\infty, 0)$ and $(2, \infty)$, $f'(x) < 0$ as e^{-x} is always positive.

Hence, f is decreasing on $(-\infty, 0)$ and $(2, \infty)$

In interval $(0, 2)$, $f'(x) > 0$

Hence, f is strictly increasing in $(0, 2)$

Thus, the correct option is **D**.

EXERCISE 6.3

Question 1:

Find the slope of the tangent to the curve $y = 3x^4 - 4x$ at $x = 4$.

Solution:

The given curve is $y = 3x^4 - 4x$

Then, the slope of the tangent to the given curve at $x = 4$ is given by,

$$\begin{aligned}
\left.\frac{dy}{dx}\right]_{x=4} &= \frac{d}{dx}\left(3x^4 - 4x\right)\Big]_{x=4} \\
&= 12x^3 - 4\Big]_{x=4} \\
&= 12(4)^3 - 4 \\
&= 12(64) - 4 \\
&= 764
\end{aligned}$$

Question 2:

Find the slope of the tangent to the curve $y = \frac{x-1}{x-2}, x \neq 2$ at $x = 10$.

Solution:

The given curve is $y = \frac{x-1}{x-2}$

Therefore,

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{x-1}{x-2}\right) \\
&= \frac{(x-2)(1)-(x-1)(1)}{(x-2)^2} \\
&= \frac{x-2-x+1}{(x-2)^2} \\
&= \frac{-1}{(x-2)^2}
\end{aligned}$$

Now, the slope of the tangent to the given curve at $x = 10$ is given by,

$$\left.\frac{dy}{dx}\right]_{x=10} = \left.\frac{-1}{(x-2)^2}\right]_{x=10}$$

$$= \frac{-1}{(10-2)^2}$$

$$= \frac{-1}{64}$$

Question 3:

Find the slope of the tangent to curve $y = x^3 - x + 1$ at the point whose x-coordinate is 2.

Solution:

The given curve is $y = x^3 - x + 1$
Therefore,

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^3 - x + 1\right)$$

$$= 3x^2 - 1$$

Now, the slope of the tangent at the point where the x-coordinate is 2 is given by,

$$\left.\frac{dy}{dx}\right]_{x=2} = 3x^2 - 1\Big]_{x=2}$$

$$= 3(2)^2 - 1$$

$$= 12 - 1$$

$$= 11$$

Question 4:

Find the slope of the tangent to curve $y = x^3 - 3x + 2$ at the point whose x-coordinate is 3.

Solution:

The given curve is $y = x^3 - 3x + 2$
Therefore,

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^3 - 3x + 2\right)$$

$$= 3x^2 - 3$$

Now, the slope of the tangent at the point where the x-coordinate is 3 is given by,

$$\left.\frac{dy}{dx}\right]_{x=2} = 3x^2 - 3\Big]_{x=3}$$
$$= 3(3)^2 - 3$$
$$= 27 - 3$$
$$= 24$$

Question 5:

Find the slope of the normal to the curve $x = a\cos^3\theta, y = a\sin^3\theta$ at $\theta = \frac{\pi}{4}$.

Solution:

The given curve is $x = a\cos^3\theta$ and $y = a\sin^3\theta$
Therefore,

$$\frac{dx}{d\theta} = \frac{d}{d\theta}\left(a\cos^3\theta\right)$$
$$= -3a\cos^2\theta\sin\theta$$
$$\frac{dy}{d\theta} = \frac{d}{d\theta}\left(a\sin^3\theta\right)$$
$$= 3a\sin^2\theta\left(\cos\theta\right)$$

Hence,

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta}$$
$$= -\frac{\sin\theta}{\cos\theta}$$
$$= -\tan\theta$$

Now, the slope of the tangent to a curve at $\theta = \frac{\pi}{4}$ is given by,

$$\left.\frac{dy}{dx}\right]_{\theta=\frac{\pi}{4}} = -\tan\theta\Big]_{\theta=\frac{\pi}{4}}$$
$$= -\tan\frac{\pi}{4}$$
$$= -1$$

Hence, the slope of the normal at $\theta = \frac{\pi}{4}$ is given by,

$$\frac{-1}{\text{slope of the tangent at } \theta = \frac{\pi}{4}} = \frac{-1}{-1} = 1$$

Question 6:

Find the slope of the normal to the curve $x = 1 - a\sin\theta$ and $y = b\cos^2\theta$ at $\theta = \frac{\pi}{2}$.

Solution:

It is given that $x = 1 - a\sin\theta$ and $y = b\cos^2\theta$

Therefore,

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(1 - a\sin\theta)$$
$$= -a\cos\theta$$
$$\frac{dy}{d\theta} = \frac{d}{d\theta}(b\cos^2\theta)$$
$$= -2b\sin\theta\cos\theta$$

Hence,

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{-2b\sin\theta\cos\theta}{-a\cos\theta}$$
$$= \frac{2b}{a}\sin\theta$$

Now, the slope of the tangent at $\theta = \frac{\pi}{2}$ is given by,

$$\left.\frac{dy}{dx}\right]_{\theta=\frac{\pi}{2}} = \left.\frac{2b}{a}\sin\theta\right]_{\theta=\frac{\pi}{2}}$$
$$= \frac{2b}{a}\sin\frac{\pi}{2}$$
$$= \frac{2b}{a}$$

Hence, the slope of normal at $\theta = \frac{\pi}{2}$ is given by,

$$\frac{-1}{\text{slope of the tangent at } \theta = \frac{\pi}{2}} = \frac{-1}{\left(\frac{2b}{a}\right)} = -\frac{a}{2b}$$

Question 7:

Find the points at which tangent to the curve $y = x^3 - 3x^2 - 9x + 7$ is parallel to the x-axis.

Solution:

The given curve is $y = x^3 - 3x^2 - 9x + 7$
Therefore,

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^3 - 3x^2 - 9x + 7\right)$$
$$= 3x^2 - 6x - 9$$

Since tangent is parallel to the x-axis if the slope of the tangent is zero.

Hence,

$$3x^2 - 6x - 9 = 0$$
$$\Rightarrow x^2 - 2x - 3 = 0$$
$$\Rightarrow (x-3)(x+1) = 0$$
$$\Rightarrow x = 3 \text{ or } x = -1$$

When, $x = 3$

Then,

$$y = (3)^3 - 3(3)^2 - 9(3) + 7$$
$$= 27 - 27 - 27 + 7$$
$$= -20$$

When, $x = -1$
Then,

$$y = (-1)^3 - 3(-1)^2 - 9(-1) + 7$$
$$= -1 - 3 + 9 + 7$$
$$= 12$$

Thus, the points at which the tangent is parallel to the x-axis are $(3, -20)$ and $(-1, 12)$.

Question 8:

Find a point on the curve $y=(x-2)^2$ at which the tangent is parallel to the chord joining the points $(2,0)$ and $(4,4)$.

Solution:

If a tangent is parallel to the chord joining the points $(2,0)$ and $(4,4)$
Then, the slope of the tangent = the slope of the chord.

Hence, the slope of chord is $\frac{4-0}{4-2}=\frac{4}{2}=2$

Now, the slope of the tangent to the given curve is,

$$\frac{dy}{dx}=2(x-2)$$

Since, the slope of the tangent = the slope of the chord.
Hence,

$$\begin{aligned}&2(x-2)=2\\&\Rightarrow x-2=1\\&\Rightarrow x=3\end{aligned}$$

When, $x=3$
Then,

$$\begin{aligned}y&=(3-2)^2\\&=1\end{aligned}$$

Hence, the point on the curve is $(3,1)$.

Question 9:

Find the point on the curve $y=x^3-11x+5$ at which the tangent is $y=x-11$.

Solution:

The given curve is $y=x^3-11x+5$

The equation of the tangent to the given curve is $y=x-11$

Therefore, slope of the tangent is 1.

Now, the slope of the tangent to the given curve is,

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^3 - 11x + 5\right)$$
$$= 3x^2 - 11$$

Hence,

$$3x^2 - 11 = 1$$
$$\Rightarrow 3x^2 = 12$$
$$\Rightarrow x^2 = 4$$
$$\Rightarrow x = \pm 2$$

When, $x = 2$
Then,

$$y = (2)^3 - 11(2) + 5$$
$$= 8 - 22 + 5$$
$$= -9$$

When, $x = -2$
Then,

$$y = (-2)^3 - 11(-2) + 5$$
$$= -8 + 22 + 5$$
$$= 19$$

Thus, the points are $(2, -9)$ and $(-2, 19)$.

Question 10:

Find the equation of all lines having slope -1 that are tangents to the curve $y = \frac{1}{x-1}, x \neq 1$.

Solution:

The given curve is $y = \frac{1}{x-1}, x \neq 1$

The slope of the tangents to the given curve is given by,

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{1}{x-1}\right)$$
$$=\frac{-1}{(x-1)^2}$$

The slope of the tangent is -1
So, we have:

$$\frac{-1}{(x-1)^2}=-1$$
$$\Rightarrow (x-1)^2=1$$
$$\Rightarrow x-1=\pm 1$$
$$\Rightarrow x=2,0$$

When, $x=0,\Rightarrow y=-1$ and when $x=2,\Rightarrow y=1$

Thus, there are two tangents to the given curve having slope -1 and passing through the points $(0,-1)$ and $(2,1)$.

Hence, the equation of the tangent through $(0,-1)$ is given by,

$$y-(-1)=-1(x-0)$$
$$\Rightarrow y+1=-x$$
$$\Rightarrow y+x+1=0$$

The equation of the tangent through $(2,1)$ is given by,

$$y-1=-1(x-2)$$
$$\Rightarrow y-1=x+2$$
$$\Rightarrow y+x-3=0$$

Thus, the equations of the required lines are $y+x+1=0$ and $y+x-3=0$.

Question 11:

Find the equations of all lines having slope 2 which are tangent to the curve $y=\frac{1}{x-3}, x\neq 3$.

Solution:

The given curve is $y=\frac{1}{x-3}$
The slope of the tangents to the given curve is given by,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{x-3}\right)$$
$$= \frac{-1}{(x-3)^2}$$

The slope of the tangent is 2
So, we have:

$$\frac{-1}{(x-3)^2} = 2$$
$$\Rightarrow 2(x-3)^2 = -1$$
$$\Rightarrow (x-3)^2 = \frac{-1}{2}$$

It is not possible since the LHS is positive and RHS is negative

Thus, there is no tangent to the curve of the slope 2.

Question 12:

Find the equations of all lines having slope 0 which are tangent to the curve $y = \frac{1}{x^2 - 2x + 3}$.

Solution:

The given curve is $y = \frac{1}{x^2 - 2x + 3}$

The slope of the tangents to the given curve is given by,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{x^2 - 2x + 3}\right)$$
$$= \frac{-(2x-2)}{(x^2 - 2x + 3)^2}$$
$$= \frac{-2(x-1)}{(x^2 - 2x + 3)^2}$$

The slope of the tangent is 0
So, we have:

$$\frac{-2(x-1)}{(x^2 - 2x + 3)^2} = 0$$
$$\Rightarrow -2(x-1) = 0$$
$$\Rightarrow x = 1$$

When, $x=1$
Then,

$$y=\frac{1}{1-2+3}$$
$$=\frac{1}{2}$$

Hence, the equation of the tangent through $\left(1,\frac{1}{2}\right)$ is given by,

$$y-\frac{1}{2}=0(x-1)$$
$$\Rightarrow y-\frac{1}{2}=0$$
$$\Rightarrow y=\frac{1}{2}$$

Thus, the equation of the line is $y=\frac{1}{2}$.

Question 13:

Find the points on the curve $\frac{x^2}{9}+\frac{y^2}{16}=1$ at which the tangents are
(i) parallel to x-axis **(ii)** parallel to y-axis

Solution:

The equation of the given curve is $\frac{x^2}{9}+\frac{y^2}{16}=1$

On differentiating both sides with respect to x, we have:

$$\Rightarrow \frac{2x}{9}+\frac{2y}{16}.\frac{dy}{dx}=0$$
$$\Rightarrow \frac{dy}{dx}=\frac{-16x}{9y}$$

(i) The tangent is parallel to the x-axis if the slope of the tangent is $\frac{-16x}{9y}=0$, which is possible if $x=0$

Thus,

$$\frac{x^2}{9}+\frac{y^2}{16}=1 \text{ for } x=0$$

Therefore,

$$\Rightarrow \frac{y^2}{16}=1$$
$$\Rightarrow y^2=16$$
$$\Rightarrow y=\pm 4$$

Hence, the points are $(0,4)$ and $(0,-4)$.

(ii) The tangent is parallel to the y-axis if the slope of the normal is 0, which gives

$$\frac{-1}{\left(\frac{-16x}{9y}\right)}=0$$
$$\Rightarrow \frac{9y}{16x}=0$$
$$\Rightarrow y=0$$

Thus,

$$\frac{x^2}{9}+\frac{y^2}{16}=1 \text{ for } y=0$$

Therefore,

$$\Rightarrow \frac{x^2}{9}=1$$
$$\Rightarrow x^2=9$$
$$\Rightarrow x=\pm 3$$

Hence, the points are $(3,0)$ and $(-3,0)$.

Question 14:

Find the equation of the tangents and normal to the given curves at the indicated points

(i) $y=x^4-6x^3+13x^2-10x+5$ at $(0,5)$

(ii) $y=x^4-6x^3+13x^2-10x+5$ at $(1,3)$

(iii) $y=x^3$ at $(1,1)$

(iv) $y=x^2$ at $(0,0)$

(v) $x=\cos t, y=\sin t$ at $t=\frac{\pi}{4}$

Solution:

(i) The equation of the curve is $y=x^4-6x^3+13x^2-10x+5$

On differentiating with respect to x, we get:

$$\frac{dy}{dx}=4x^3-18x^2+26x-10$$
$$\left.\frac{dy}{dx}\right]_{(0,5)}=-10$$

Thus, the slope of the tangent at $(0,5)$ is -10.

The equation of the tangent is given as:

$$y-5=-10(x-0)$$
$$\Rightarrow y-5=-10x$$
$$\Rightarrow 10x+y=5$$

Slope of normal at $(0,5)$ is

$$\frac{-1}{\text{slope of the tangent at }(0,5)}=\frac{-1}{-10}=\frac{1}{10}$$

Therefore, the equation of the normal at $(0,5)$ is given as:

$$y-5=\frac{1}{10}(x-0)$$
$$\Rightarrow 10y-50=x$$
$$\Rightarrow x-10y+50=0$$

(ii) The equation of the curve is $y=x^4-6x^3+13x^2-10x+5$ at $(1,3)$

On differentiating with respect to x, we get:

$$\frac{dy}{dx}=4x^3-18x^2+26x-10$$
$$\left.\frac{dy}{dx}\right]_{(1,3)}=4-18+26-10=2$$

Thus, the slope of the tangent at $(1,3)$ is 2.

The equation of the tangent is given as:

$$y-3=2(x-1)$$
$$\Rightarrow y-3=2x-2$$
$$\Rightarrow y=2x+1$$

Slope of normal at $(1,3)$ is

$$\frac{-1}{\text{slope of the tangent at }(1,3)}=\frac{-1}{2}$$

Therefore, the equation of the normal at $(1,3)$ is given as:

$$y-3=\frac{1}{2}(x-1)$$
$$\Rightarrow 2y-6=x+1$$
$$\Rightarrow x+2y-7=0$$

(iii) The equation of the curve is $y=x^3$ at $(1,1)$

On differentiating with respect to x, we get:

$$\frac{dy}{dx}=3x^2$$
$$\left.\frac{dy}{dx}\right]_{(1,1)}=3(1)^2=3$$

Thus, the slope of the tangent at $(1,1)$ is 3.

The equation of the tangent is given as:

$$y-1=3(x-1)$$
$$\Rightarrow y=3x-2$$

Slope of normal at $(1,1)$ is

$$\frac{-1}{\text{slope of the tangent at }(1,1)}=\frac{-1}{3}$$

Therefore, the equation of the normal at $(1,1)$ is given as:

$$y-1=\frac{-1}{3}(x-1)$$
$$\Rightarrow 3y-3=-x+1$$
$$\Rightarrow x+3y-4=0$$

(iv) The equation of the curve is $y = x^2$ at $(0,0)$

On differentiating with respect to x, we get:

$$\frac{dy}{dx} = 2x$$

$$\left.\frac{dy}{dx}\right]_{(0,0)} = 0$$

Thus, the slope of the tangent at $(0,0)$ is 0.

The equation of the tangent is given as:

$$y - 0 = 0(x - 0)$$
$$\Rightarrow y = 0$$

Slope of normal at $(0,0)$ is

$\frac{-1}{\text{slope of the tangent at } (0,0)} = \frac{-1}{0}$, which is not defined.

Therefore, the equation of the normal at $(0,0)$ is given as $x = 0$

The equation of the curve is $x = \cos t$ and $y = \sin t$ at $t = \frac{\pi}{4}$

On differentiating with respect to t, we get:

$\frac{dx}{dt} = -\sin t$ and $\frac{dy}{dt} = \cos t$

Therefore,

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\cos t}{-\sin t} = -\cot t$$

Hence,

$$\left.\frac{dy}{dx}\right]_{t=\frac{\pi}{4}} = -\cot t = -1$$

Thus, the slope of the tangent at $t = \frac{\pi}{4}$ is -1.
Hence,

$x=\frac{1}{\sqrt{2}}$ and $y=\frac{1}{\sqrt{2}}$

The equation of the tangent is given as:

$$\Rightarrow y-\frac{1}{\sqrt{2}}=-1\left(x-\frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow x+y-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}=0$$

$$\Rightarrow x+y-\sqrt{2}=0$$

Slope of normal at $t=\frac{\pi}{4}$ is

$$\frac{-1}{\text{slope of the tangent at } t=\frac{\pi}{4}}=\frac{-1}{-1}=1$$

Therefore, the equation of the normal at $t=\frac{\pi}{4}$ is given as:

$$y-\frac{1}{\sqrt{2}}=1\left(x-\frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow x=y$$

Question 15:

Find the equation of the tangent line to the curve $y=x^2-2x+7$ which is

(a) parallel to the line $2x-y+9=0$

(b) perpendicular to the line $5y-15x=13$

Solution:

The equation of the curve is $y=x^2-2x+7$
On differentiating with respect to x, we get:

$$\frac{dy}{dx}=2x-2$$

(a) The equation of the line is $2x-y+9=0$

$$\Rightarrow y=2x+9$$

This is of the form $y=mx+c$

Hence, the slope of line is 2

If a tangent is parallel to the line $2x - y + 9 = 0$, then the slope of the tangent is equal to the slope of the line.

Therefore, we have:

$$2 = 2x - 2$$
$$\Rightarrow 2x = 4$$
$$\Rightarrow x = 2$$

Now, $x = 2$
Then,

$$\Rightarrow y = 4 - 4 + 7$$
$$\Rightarrow y = 7$$

Thus, the equation of tangent passing through $(2,7)$ is given by,

$$y - 7 = 2(x - 2)$$
$$\Rightarrow y - 2x - 3 = 0$$

(b) perpendicular to the line $5y - 15x = 13$

$$\Rightarrow y = 3x + \frac{13}{5}$$

This is of the form $y = mx + c$

Hence, the slope of line is 3

If a tangent is perpendicular to the line $5y - 15x = 13$, then the slope of the tangent is,

$$\frac{-1}{\text{slope of the line}} = \frac{-1}{3}$$

Therefore, we have:

$$2x-2=\frac{-1}{3}$$
$$\Rightarrow 2x=\frac{-1}{3}+2$$
$$\Rightarrow 2x=\frac{5}{3}$$
$$\Rightarrow x=\frac{5}{6}$$

Now, $x=\frac{5}{6}$

Then,

$$y=\frac{25}{36}+\frac{10}{6}+7$$
$$=\frac{25-60+252}{36}$$
$$=\frac{217}{36}$$

Thus, the equation of tangent passing through $\left(\frac{5}{6},\frac{217}{36}\right)$ is given by,

$$y-\frac{217}{36}=\frac{1}{3}\left(x-\frac{5}{6}\right)$$
$$\Rightarrow \frac{36y-217}{36}=\frac{-1}{18}(6x-5)$$
$$\Rightarrow 36y-217=-2(6x-5)$$
$$\Rightarrow 36y-217=-12x+10$$
$$\Rightarrow 36y+12x-227=0$$

Question 16:

Show that the tangents to the curve $y=7x^3+11$ at the points where $x=2$ and $x=-2$ are parallel.

Solution:

The equation of the given curve is $y=7x^3+11$.
Therefore,

$$\frac{dy}{dx}=21x^2$$

Thus, the slope of the tangent at the point where $x = 2$ is given by,

$$\left.\frac{dy}{dx}\right]_{x=2} = 21(2)^2 = 84$$

Also, the slope of the tangent at the point where $x = -2$ is given by,

$$\left.\frac{dy}{dx}\right]_{x=-2} = 21(-2)^2 = 84$$

It is observed clearly that the slopes of the tangents at the points where $x = 2$ and $x = -2$ are equal.

Hence, the two tangents are parallel.

Question 17:

Find the points on the curve $y = x^3$ at which the slope of the tangent is equal to the *y*-coordinates of the point.

Solution:

The equation of the given curve is $y = x^3$
Therefore,

$$\frac{dy}{dx} = 3x^2$$

When the slope of the tangent is equal to the *y*-coordinate of the point, then According to the question, $y = 3x^2$

Also, we have $y = x^3$

Therefore,

$$3x^2 = x^3$$
$$\Rightarrow x^2(x-3) = 0$$
$$\Rightarrow x = 0, x = 3$$

When, $x = 0, \Rightarrow y = 3$ and $x = 3, \Rightarrow y = 3(3)^2 = 27$

Thus, the points are $(0,0)$ and $(3,27)$.

Question 18:

For the curve $y=4x^3-2x^5$, find all the points at which the tangent passes through the origin.

Solution:

The equation of the given curve is $y=4x^3-2x^5$
Therefore,

$$\frac{dy}{dx}=12x^2-10x^4$$

Hence, the slope of the tangent at the point (x,y) is $12x^2-10x^4$

Thus, the equation of the tangent at (x,y) is given by,

$$Y-y=\left(12x^2-10x^4\right)(X-x)$$

When the tangent passes through the origin $(0,0)$, $X=Y=0$.
Therefore,

$$-y=\left(12x^2-10x^4\right)(-x)$$
$$y=12x^3-10x^5$$

Also, we have $y=4x^3-2x^5$
Hence,

$$\therefore 12x^3-10x^5=4x^3-2x^5$$
$$\Rightarrow 8x^5-8x^3=0$$
$$\Rightarrow x^5-x^3=0$$
$$\Rightarrow x^3\left(x^2-1\right)=0$$
$$\Rightarrow x=0,\pm1$$

When, $x=0,\Rightarrow y=4(0)^3-2(0)^5=0$

When, $x=1,\Rightarrow y=4(1)^3-2(1)^5=2$

When, $x=-1,\Rightarrow y=4(-1)^3-2(-1)^5=-2$

Thus, the points are $(0,0)$, $(1,2)$ and $(-1,-2)$

Question 19:

Find the points on the curve $x^2 + y^2 - 2x - 3 = 0$ at which the tangents are parallel to the x-axis.

Solution:

The equation of the given curve is $x^2 + y^2 - 2x - 3 = 0$

On differentiating with respect to x, we have:

$$2x + 2y\frac{dy}{dx} - 2 = 0$$
$$\Rightarrow y\frac{dy}{dx} = 1 - x$$
$$\Rightarrow \frac{dy}{dx} = \frac{1-x}{y}$$

Now, the tangents are parallel to the x-axis if the slope of the tangent is 0.
Therefore,

$$\frac{1-x}{y} = 0$$
$$\Rightarrow 1 - x = 0$$
$$\Rightarrow x = 1$$

But we have $x^2 + y^2 - 2x - 3 = 0$ for $x = 1$
Hence,

$$\Rightarrow y^2 = 4$$
$$\Rightarrow y = \pm 2$$

Thus, the points are $(1, 2)$ and $(1, -2)$.

Question 20:

Find the equation of the normal at the point (am^2, am^3) for the curve $ay^2 = x^3$.

Solution:

The equation of the given curve is $ay^2 = x^3$

On differentiating with respect to x, we have:

$$2ay\frac{dy}{dx}=3x^2$$
$$\Rightarrow \frac{dy}{dx}=\frac{3x^2}{2ay}$$

The slope of a tangent to the curve at $\left(am^2, am^3\right)$ is

$$\left.\frac{dy}{dx}\right]_{\left(am^2, am^3\right)}=\frac{3\left(am^2\right)^2}{2a\left(am^3\right)}$$
$$=\frac{3a^2m^4}{2a^2m^3}$$
$$=\frac{3m}{2}$$

Therefore, the slope of normal at $\left(am^2, am^3\right)$ is

$$\frac{-1}{\text{slope of the tangent at }\left(am^2, am^3\right)}=\frac{-1}{\left(\frac{3m}{2}\right)}=\frac{-2}{3m}$$

Hence, the equation of the normal at $\left(am^2, am^3\right)$ is given by,

$$y-am^3=\frac{-2}{3m}\left(x-am^2\right)$$
$$\Rightarrow 3my-3am^4=-2x+2am^2$$
$$\Rightarrow 2x+3my-am^2\left(2+3m^2\right)=0$$

Question 21:

Find the equation of the normal to the curve $y=x^3+2x+6$ which are parallel to the line $x+14y+4=0$.

Solution:

The equation of the given curve is $y=x^3+2x+6$

The slope of the tangent to the given curve at any point (x, y) is given by,

$$\frac{dy}{dx}=3x^2+2$$

Therefore, slope of the normal to the given curve is,

$$\frac{-1}{\text{slope of the tangent}}=\frac{-1}{3x^2+2}$$

The equation of the given line is $x+14y+4=0$

$$\Rightarrow y=-\frac{1}{14}x-\frac{4}{14}$$, which is the form of $y=mx+c$

Hence,

$$\frac{-1}{3x^2+2}=\frac{-1}{14}$$
$$\Rightarrow 3x^2+2=14$$
$$\Rightarrow 3x^2=12$$
$$\Rightarrow x^2=4$$
$$\Rightarrow x=\pm 2$$

When, $x=2,\Rightarrow y=8+4+6=18$
When, $x=-2,\Rightarrow y=-8-4+6=-6$

Therefore, there are two normal to the given curve with slope $\frac{-1}{14}$ and passing through the points $(2,18)$ and $(-2,-6)$.

Thus, the equation of the normal through $(2,18)$ is

$$y-18=\frac{-1}{14}(x-2)$$
$$\Rightarrow 14y-252=x+2$$
$$\Rightarrow x+14y-254=0$$

And the equation of the normal through $(-2,-6)$ is

$$y-(-6)=\frac{-1}{14}(x-(-2))$$
$$\Rightarrow 14y+84=-x-2$$
$$\Rightarrow x+14y+86=0$$

Hence, the equations of the normal to the given curve are $x+14y-254=0$ and $x+14y+86=0$.

Question 22:

Find the equation of the tangent and normal to the parabola $y^2=4ax$ at the point $(at^2, 2at)$.

Solution:

The equation of the given parabola is $y^2 = 4ax$.

On differentiating both sides with respect to x, we have:

$$2y\frac{dy}{dx} = 4a$$
$$\Rightarrow \frac{dy}{dx} = \frac{2a}{y}$$

Therefore, the slope of the tangent at $(at^2, 2at)$ is

$$\left.\frac{dy}{dx}\right]_{(at^2, 2at)} = \frac{2a}{2at} = \frac{1}{t}$$

Hence, the equation of the tangent at $(at^2, 2at)$ is

$$y - 2at = \frac{1}{t}\left(x - at^2\right)$$
$$\Rightarrow ty - 2at^2 = x - at^2$$
$$\Rightarrow ty = x + at^2$$

Now, the slope of the normal at $(at^2, 2at)$ is

$$\frac{-1}{\text{slope of the tangent at } \left(at^2, 2at\right)} = \frac{-1}{\left(\frac{1}{t}\right)} = -t$$

Thus, the equation of the normal at $(at^2, 2at)$ is

$$y - 2at = -t\left(x - at^2\right)$$
$$\Rightarrow y - 2at = -tx + at^3$$
$$\Rightarrow y = -tx + 2at + at^3$$

Question 23:

Prove that the curves $x = y^2$ and $xy = k$ cut at right angles if $8k^2 = 1$.
[**Hint:** Two curves intersect at right angle if the tangents to the curve at the point of intersection are perpendicular to each other.]

Solution:

The equations of the given curves are $x = y^2$ and $xy = k$

Putting $x=y^2$ in $xy=k$

$$y^3=k$$

$$\Rightarrow y=k^{\frac{1}{3}}$$

$$\Rightarrow x=k^{\frac{2}{3}}$$

Thus, the point of intersection of the given curves is $\left(k^{\frac{2}{3}},k^{\frac{1}{3}}\right)$.

Differentiating $x=y^2$ with respect to x, we have:

$$1=2y\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{2y}$$

Therefore, the slope of the tangent to the curve $x=y^2$ at $\left(k^{\frac{2}{3}},k^{\frac{1}{3}}\right)$ is

$$\left.\frac{dy}{dx}\right]_{\left(k^{\frac{2}{3}},k^{\frac{1}{3}}\right)}=\frac{1}{2k^{\frac{1}{3}}}$$

On differentiating $xy=k$ with respect to x, we have:

$$x\frac{dy}{dx}+y=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{-y}{x}$$

Now, slope of the tangent to the curve $xy=k$ at $\left(k^{\frac{2}{3}},k^{\frac{1}{3}}\right)$ is

$$\left.\frac{dy}{dx}\right]_{\left(k^{\frac{2}{3}},k^{\frac{1}{3}}\right)}=\left.\frac{-y}{x}\right]_{\left(k^{\frac{2}{3}},k^{\frac{1}{3}}\right)}$$

$$=-\frac{k^{\frac{1}{3}}}{k^{\frac{2}{3}}}$$

$$=\frac{-1}{k^{\frac{1}{3}}}$$

We know that two curves intersect at right angles if the tangents to the curves at the point of intersection i.e., at $\left(k^{\frac{2}{3}},k^{\frac{1}{3}}\right)$ are perpendicular to each other.

This implies that we should have the product of the tangents as -1.
Thus, the given two curves cut at right angles if the product of the slopes of their respective tangents at $\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)$ is -1. i.e.,

$$\left(\frac{1}{2k^{\frac{1}{3}}}\right)\left(\frac{-1}{k^{\frac{1}{3}}}\right) = -1$$
$$\Rightarrow 2k^{\frac{2}{3}} = 1$$
$$\Rightarrow \left(2k^{\frac{2}{3}}\right)^3 = (1)^3$$
$$\Rightarrow 8k^2 = 1$$

Hence, the given curves cut at right angle if $8k^2 = 1$.

Question 24:

Find the equation of the tangent and normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_0, y_0).

Solution:

Differentiating $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with respect to x, we have:

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$
$$\Rightarrow \frac{2y}{b^2}\frac{dy}{dx} = \frac{2x}{a^2}$$
$$\Rightarrow \frac{dy}{dx} = \frac{b^2 x}{a^2 y}$$

Therefore, the slope of the tangent at (x_0, y_0) is

$$\left.\frac{dy}{dx}\right]_{(x_0, y_0)} = \frac{b^2 x_0}{a^2 y_0}$$

Hence, the equation of tangent at (x_0, y_0) is

$$y - y_0 = \frac{b^2 x_0}{a^2 y_0}(x - x_0)$$

$$\Rightarrow a^2 y y_0 - a^2 y_0^{\ 2} = b^2 x x_0 - b^2 x_0^{\ 2}$$

$$\Rightarrow b^2 x x_0 - a^2 y y_0 - b^2 x_0^{\ 2} - a^2 y_0^{\ 2} = 0$$

$$\Rightarrow \frac{x x_0}{a^2} - \frac{y y_0}{b^2} - \left(\frac{x_0^{\ 2}}{a^2} - \frac{y_0^{\ 2}}{b^2}\right) = 0$$

$$\Rightarrow \frac{x x_0}{a^2} - \frac{y y_0}{b^2} - 1 = 0$$

$$\Rightarrow \frac{x x_0}{a^2} - \frac{y y_0}{b^2} = 1$$

Now, the slope of normal at (x_0, y_0) is

$$\frac{-1}{\text{slope of the tangent at } (x_0, y_0)} = \frac{-a^2 y_0}{b^2 x_0}$$

Hence, the equation of normal at (x_0, y_0) is

$$y - y_0 = \frac{-a^2 y_0}{b^2 x_0}(x - x_0)$$

$$\Rightarrow \frac{y - y_0}{a^2 y_0} = \frac{-(x - x_0)}{b^2 x_0}$$

$$\Rightarrow \frac{y - y_0}{a^2 y_0} - \frac{(x - x_0)}{b^2 x_0} = 0$$

Question 25:

Find the equation of the tangent to the curve $y = \sqrt{3x - 2}$ which is parallel to the line $4x - 2y + 5 = 0$.

Solution:

The equation of the given curve is $y = \sqrt{3x - 2}$

The slope of the tangent to the given curve at any point (x, y) is given by,

$$\frac{dy}{dx} = \frac{3}{2\sqrt{3x - 2}}$$

The equation of the given line is $4x - 2y + 5 = 0$.

$\Rightarrow y = 2x + \frac{5}{2}$, which is the form of $y = mx + c$

Hence, the slope of line is 2.

Now, the tangent to the given curve is parallel to the line $4x-2y+5=0$ if the slope of the tangent is equal to the slope of the line.

Therefore,

$$\frac{3}{2\sqrt{3x-2}}=2$$
$$\Rightarrow \sqrt{3x-2}=\frac{3}{4}$$
$$\Rightarrow 3x-2=\frac{9}{16}$$
$$\Rightarrow 3x=\frac{9}{16}+2=\frac{41}{16}$$
$$\Rightarrow x=\frac{41}{48}$$

When, $x=\frac{41}{48}$

Then,

$$y=\sqrt{3\left(\frac{41}{48}\right)-2}$$
$$=\sqrt{\frac{41}{16}-2}$$
$$=\sqrt{\frac{41-32}{16}}$$
$$=\sqrt{\frac{9}{16}}$$
$$=\frac{3}{4}$$

Thus, equation of the tangent passing through the point $\left(\frac{41}{48},\frac{3}{4}\right)$ is

$$y-\frac{3}{4}=2\left(x-\frac{41}{48}\right)$$
$$\Rightarrow \frac{4y-3}{4}=2\left(\frac{48x-41}{48}\right)$$
$$\Rightarrow 4y-3=\left(\frac{48x-41}{6}\right)$$
$$\Rightarrow 24y-18=48x-41$$
$$\Rightarrow 48x-24y=23$$

Question 26:

The slope of the normal to the curve $y=2x^2+3\sin x$ at $x=0$ is

(A) 3 **(B)** $\frac{1}{3}$ **(C)** -3 **(D)** $-\frac{1}{3}$

Solution:

The equation of the given curve is $y=2x^2+3\sin x$.

Slope of the tangent to the given curve at $x=0$ is given by,

$$\left.\frac{dy}{dx}\right]_{x=0}=4x+3\cos x\Big]_{x=0}$$
$$=0+3\cos 0$$
$$=3$$

Hence, the slope of the normal to the given curve at $x=0$ is

$$\frac{-1}{\text{slope of the tangent at }(x=0)}=\frac{-1}{3}$$

Thus, the correct option is **D**.

Question 27:

The line $y=x+1$ is a tangent to the curve $y^2=4x$ at the point

(A) $(1,2)$ **(B)** $(2,1)$ **(C)** $(1,-2)$ **(D)** $(-1,2)$

Solution:

The equation of the given curve is $y^2=4x$.

Differentiating with respect to x, we have:

$$2y\frac{dy}{dx}=4$$
$$\Rightarrow \frac{dy}{dx}=\frac{2}{y}$$

The given line is $y=x+1$ which is of the form $y=mx+c$.

Hence, slope of the line is 1

The line $y=x+1$ is a tangent to the given curve if the slope of the line is equal to the slope of the tangent.
Also, the line must intersect the curve.

Thus, we must have:

$$\frac{2}{y}=1$$
$$\Rightarrow y=2$$

Therefore,

$$y=x+1$$
$$\Rightarrow x=y-1$$
$$\Rightarrow x=2-1$$
$$\Rightarrow x=1$$

Hence, the line $y=x+1$ is a tangent to the given curve at the point $(1,2)$.

Thus, the correct answer is **A**.

EXERCISE 6.4

Question 1:

Using differentials, find the approximate value of each of the following up to 3 places of decimals.

(i) $\sqrt{25.3}$ (ii) $\sqrt{49.5}$ (iii) $\sqrt{0.6}$

(iv) $(0.009)^{\frac{1}{3}}$ (v) $(0.999)^{\frac{1}{10}}$ (vi) $(15)^{\frac{1}{4}}$

(vii) $(26)^{\frac{1}{3}}$ (viii) $(255)^{\frac{1}{4}}$ (ix) $(82)^{\frac{1}{4}}$

(x) $(401)^{\frac{1}{2}}$ (xi) $(0.0037)^{\frac{1}{2}}$ (xii) $(26.57)^{\frac{1}{3}}$

(xiii) $(81.5)^{\frac{1}{4}}$ (xiv) $(3.968)^{\frac{3}{2}}$ (xv) $(32.15)^{\frac{1}{5}}$

Solution:

(i) $\sqrt{25.3}$

Consider $y=\sqrt{x}$.

Let $x=25$ and $\Delta x=0.3$

Then,

$$\begin{aligned}\Delta y &= \sqrt{x+\Delta x}-\sqrt{x}\\ &= \sqrt{25.3}-\sqrt{25}\\ &= \sqrt{25.3}-5\\ \Delta y+5 &= \sqrt{25.3}\end{aligned}$$

Now, dy is approximately equal to Δy and is given by,

$$\begin{aligned}dy &= \left(\frac{dy}{dx}\right)\Delta x\\ &= \frac{1}{2\sqrt{x}}(0.3) \qquad \left[\because y=\sqrt{x}\right]\\ &= \frac{1}{2\sqrt{25}}(0.3)\\ &= 0.03\end{aligned}$$

Hence,

$$\sqrt{25.3} = 0.03 + 5$$
$$= 5.03$$

Thus, the approximate value of $\sqrt{25.3} = 5.03$.

(ii) $\sqrt{49.5}$

Consider $y = \sqrt{x}$

Let $x = 49$ and $\Delta x = 0.5$

Then,

$$\Delta y = \sqrt{x + \Delta x} - \sqrt{x}$$
$$= \sqrt{49.5} - \sqrt{49}$$
$$= \sqrt{49.5} - 7$$
$$\Delta y + 7 = \sqrt{49.5}$$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x$$
$$= \frac{1}{2\sqrt{x}}(0.5) \qquad \left[\because y = \sqrt{x}\right]$$
$$= \frac{1}{2\sqrt{49}}(0.5)$$
$$= \frac{1}{14}(0.5)$$
$$= 0.035$$

Hence,

$$\sqrt{49.5} = 7 + 0.035$$
$$= 7.035$$

Thus, the approximate value of $\sqrt{49.5} = 7.035$.

(iii) $\sqrt{0.6}$

Consider $y = \sqrt{x}$.

Let $x = 1$ and $\Delta x = -0.4$

Then,

$$\Delta y = \sqrt{x+\Delta x} - \sqrt{x}$$
$$= \sqrt{0.6} - 1$$
$$\Delta y + 1 = \sqrt{0.6}$$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x$$
$$= \frac{1}{2\sqrt{x}}(-0.4) \qquad \left[\because y = \sqrt{x}\right]$$
$$= \frac{1}{2}(-0.4)$$
$$= -0.2$$

Hence,

$$\sqrt{0.6} = 1 + (-0.2)$$
$$= 1 - 0.2$$
$$= 0.8$$

Thus, the approximate value of $\sqrt{0.6} = 0.8$.

(iv) $(0.009)^{\frac{1}{3}}$

Consider $y = (x)^{\frac{1}{3}}$

Let $x = 0.008$ and $\Delta x = 0.001$

Then,

$$\Delta y = (x+\Delta x)^{\frac{1}{3}} - (x)^{\frac{1}{3}}$$
$$= (0.009)^{\frac{1}{3}} - (0.008)^{\frac{1}{3}}$$
$$= (0.009)^{\frac{1}{3}} - 0.2$$
$$\Delta y + 0.2 = (0.009)^{\frac{1}{3}}$$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x$$
$$= \frac{1}{3(x)^{\frac{2}{3}}}(\Delta x) \qquad \left[\because y = x^{\frac{1}{3}}\right]$$
$$= \frac{1}{3\times 0.04}(0.001)$$
$$= \frac{0.001}{0.12}$$
$$= 0.008$$

Hence,

$$(0.009)^{\frac{1}{3}} = 0.2 + 0.008$$
$$= 0.208$$

Thus, the approximate value of $(0.009)^{\frac{1}{3}} = 0.208$.

(v) $(0.999)^{\frac{1}{10}}$

Consider $y = (x)^{\frac{1}{10}}$

Let $x = 1$ and $\Delta x = -0.001$

Then,

$$\Delta y = (x + \Delta x)^{\frac{1}{10}} - (x)^{\frac{1}{10}}$$
$$= (0.999)^{\frac{1}{10}} - 1$$
$$\Delta y + 1 = (0.999)^{\frac{1}{10}}$$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x$$
$$= \frac{1}{10(x)^{\frac{9}{10}}}(\Delta x) \qquad \left[\because y = (x)^{\frac{1}{10}}\right]$$
$$= \frac{1}{10}(-0.001)$$
$$= -0.0001$$

Hence,

$$(0.999)^{\frac{1}{10}} = 1 + (-0.0001)$$
$$= 0.9999$$

Thus, the approximate value of $(0.999)^{\frac{1}{10}} = 0.9999$.

(vi) $(15)^{\frac{1}{4}}$

Consider $y = (x)^{\frac{1}{4}}$

Let $x = 16$ and $\Delta x = -1$

Then,

$$\begin{aligned} \Delta y &= (x + \Delta x)^{\frac{1}{4}} - (x)^{\frac{1}{4}} \\ &= (15)^{\frac{1}{4}} - (16)^{\frac{1}{4}} \\ &= (15)^{\frac{1}{4}} - 2 \\ \Delta y + 2 &= (15)^{\frac{1}{4}} \end{aligned}$$

Now, dy is approximately equal to Δy and is given by,

$$\begin{aligned} dy &= \left(\frac{dy}{dx}\right)\Delta x \\ &= \frac{1}{4(x)^{\frac{3}{4}}}(\Delta x) \qquad \left[\because y = (x)^{\frac{1}{4}}\right] \\ &= \frac{1}{4(16)^{\frac{3}{4}}}(-1) \\ &= \frac{-1}{4 \times 8} \\ &= \frac{-1}{32} \\ &= -0.03125 \end{aligned}$$

Hence,

$$\begin{aligned} (15)^{\frac{1}{4}} &= 2 + (-0.03125) \\ &= 1.96875 \end{aligned}$$

Thus, the approximate value of $(15)^{\frac{1}{4}} = 1.96875$.

(vii) $(26)^{\frac{1}{3}}$

Consider $y=(x)^{\frac{1}{3}}$.

Let $x=27$ and $\Delta x=-1$

Then,

$$\Delta y=(x+\Delta x)^{\frac{1}{3}}-(x)^{\frac{1}{3}}$$
$$=(26)^{\frac{1}{3}}-(27)^{\frac{1}{3}}$$
$$=(26)^{\frac{1}{3}}-3$$
$$\Delta y+3=(26)^{\frac{1}{3}}$$

Now, dy is approximately equal to Δy and is given by,

$$dy=\left(\frac{dy}{dx}\right)\Delta x$$
$$=\frac{1}{3(x)^{\frac{2}{3}}}(\Delta x) \qquad \left[\because\ y=(x)^{\frac{1}{3}}\right]$$
$$=\frac{1}{3(27)^{\frac{2}{3}}}(-1)$$
$$=\frac{-1}{27}$$
$$=-0.0\overline{370}$$

Hence,

$$(26)^{\frac{1}{3}}=3+(-0.0370)$$
$$=2.9629$$

Thus, the approximate value of $(26)^{\frac{1}{3}}=2.9629$.

(viii) $(255)^{\frac{1}{4}}$

Consider $y=(x)^{\frac{1}{4}}$

Let $x=256$ and $\Delta x=-1$

Then,

$$\Delta y = (x + \Delta x)^{\frac{1}{4}} - (x)^{\frac{1}{4}}$$
$$= (255)^{\frac{1}{4}} - (256)^{\frac{1}{4}}$$
$$= (255)^{\frac{1}{4}} - 4$$
$$\Delta y + 4 = (255)^{\frac{1}{4}}$$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x$$
$$= \frac{1}{4(x)^{\frac{3}{4}}}(\Delta x) \qquad \left[\because y = (x)^{\frac{1}{4}}\right]$$
$$= \frac{1}{4(256)^{\frac{3}{4}}}(-1)$$
$$= \frac{-1}{4 \times 4^3}$$
$$= \frac{-1}{32}$$
$$= -0.0039$$

Hence,

$$(255)^{\frac{1}{4}} = 4 + (-0.0039)$$
$$= 3.9961$$

Thus, the approximate value of $(255)^{\frac{1}{4}} = 3.9961$.

(ix) $(82)^{\frac{1}{4}}$

Consider $y=(x)^{\frac{1}{4}}$.

Let $x=81$ and $\Delta x=1$

Then,

$$\begin{aligned}\Delta y &= (x+\Delta x)^{\frac{1}{4}}-(x)^{\frac{1}{4}}\\ &= (82)^{\frac{1}{4}}-(81)^{\frac{1}{4}}\\ &= (82)^{\frac{1}{4}}-3\\ \Delta y+3 &= (82)^{\frac{1}{4}}\end{aligned}$$

Now, dy is approximately equal to Δy and is given by,

$$\begin{aligned}dy &= \left(\frac{dy}{dx}\right)\Delta x\\ &= \frac{1}{4(x)^{\frac{3}{4}}}(\Delta x) \qquad \left[\because y=(x)^{\frac{1}{4}}\right]\\ &= \frac{1}{4(81)^{\frac{3}{4}}}(1)\\ &= \frac{1}{4\times 3^3}\\ &= \frac{1}{108}\\ &= 0.009\end{aligned}$$

Hence,

$$\begin{aligned}(82)^{\frac{1}{4}} &= 3+0.009\\ &= 3.009\end{aligned}$$

Thus, the approximate value of $(82)^{\frac{1}{4}}=3.009$.

(x) $(401)^{\frac{1}{2}}$

Consider $y=(x)^{\frac{1}{2}}$

Let $x=400$ and $\Delta x=1$

Then,

$$\begin{aligned}\Delta y &= (x+\Delta x)^{\frac{1}{2}}-(x)^{\frac{1}{2}}\\ &= (401)^{\frac{1}{2}}-(400)^{\frac{1}{2}}\\ &= (401)^{\frac{1}{2}}-20\\ \Delta y+20 &= (401)^{\frac{1}{2}}\end{aligned}$$

Now, dy is approximately equal to Δy and is given by,

$$\begin{aligned}dy &= \left(\frac{dy}{dx}\right)\Delta x\\ &= \frac{1}{2\sqrt{x}}(\Delta x) \qquad \left[\because y=(x)^{\frac{1}{2}}\right]\\ &= \frac{1}{2(20)}(1)\\ &= \frac{1}{40}\\ &= 0.025\end{aligned}$$

Hence,

$$\begin{aligned}(401)^{\frac{1}{2}} &= 20+0.025\\ &= 20.025\end{aligned}$$

Thus, the approximate value of $(401)^{\frac{1}{2}}=20.025$.

(xi) $(0.0037)^{\frac{1}{2}}$

Consider $y=(x)^{\frac{1}{2}}$

Let $x=0.0036$ and $\Delta x=0.0001$

Then,

$$\Delta y = (x+\Delta x)^{\frac{1}{2}} - (x)^{\frac{1}{2}}$$
$$= (0.0037)^{\frac{1}{2}} - (0.0036)^{\frac{1}{2}}$$
$$= (0.0037)^{\frac{1}{2}} - 0.06$$
$$\Delta y + 0.06 = (0.0037)^{\frac{1}{2}}$$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x$$
$$= \frac{1}{2\sqrt{x}}(\Delta x) \qquad \left[\because y = (x)^{\frac{1}{2}}\right]$$
$$= \frac{1}{2(0.06)}(0.0001)$$
$$= \frac{0.0001}{0.12}$$
$$= 0.0008325$$

Hence,

$$(0.0037)^{\frac{1}{2}} = 0.06 + 0.00083$$
$$= 0.06083$$

Thus, the approximate value of $(0.0037)^{\frac{1}{2}} = 0.06083$.

(xii) $(26.57)^{\frac{1}{3}}$

Consider $y = (x)^{\frac{1}{3}}$

Let $x = 27$ and $\Delta x = -0.43$

Then,

$$\Delta y = (x+\Delta x)^{\frac{1}{3}} - (x)^{\frac{1}{3}}$$
$$= (26.57)^{\frac{1}{3}} - (27)^{\frac{1}{3}}$$
$$= (26.57)^{\frac{1}{3}} - 3$$
$$\Delta y + 3 = (26.57)^{\frac{1}{3}}$$

Now, dy is approximately equal to Δy and is given by,

$$\begin{aligned} dy &= \left(\frac{dy}{dx}\right)\Delta x \\ &= \frac{1}{3(x)^{\frac{2}{3}}}(\Delta x) \qquad \left[\because y = (x)^{\frac{1}{3}}\right] \\ &= \frac{1}{3(9)}(-0.43) \\ &= \frac{-0.43}{27} \\ &= -0.015 \end{aligned}$$

Hence,

$$\begin{aligned} (26.57)^{\frac{1}{3}} &= 3 + (-0.015) \\ &= 2.984 \end{aligned}$$

Thus, the approximate value of $(26.57)^{\frac{1}{3}} = 2.984$.

(xiii) $(81.5)^{\frac{1}{4}}$

Consider $y = (x)^{\frac{1}{4}}$

Let $x = 81$ and $\Delta x = 0.5$

Then,

$$\begin{aligned} \Delta y &= (x + \Delta x)^{\frac{1}{4}} - (x)^{\frac{1}{4}} \\ &= (81.5)^{\frac{1}{4}} - (81)^{\frac{1}{4}} \\ &= (81.5)^{\frac{1}{4}} - 3 \\ \Delta y + 3 &= (81.5)^{\frac{1}{4}} \end{aligned}$$

Now, dy is approximately equal to Δy and is given by,

$$
\begin{aligned}
dy &= \left(\frac{dy}{dx}\right)\Delta x \\
&= \frac{1}{4(x)^{\frac{3}{4}}}(\Delta x) \qquad \left[\because y=(x)^{\frac{1}{4}}\right] \\
&= \frac{1}{4(3)^{3}}(0.5) \\
&= \frac{0.5}{108} \\
&= 0.0046
\end{aligned}
$$

Hence,

$$
\begin{aligned}
(81.5)^{\frac{1}{4}} &= 3+0.0046 \\
&= 3.0046
\end{aligned}
$$

Thus, the approximate value of $(81.5)^{\frac{1}{4}} = 3.0046$.

(xiv) $(3.968)^{\frac{3}{2}}$

Consider $y=(x)^{\frac{3}{2}}$

Let $x=4$ and $\Delta x=-0.032$

Then,

$$
\begin{aligned}
\Delta y &= (x+\Delta x)^{\frac{3}{2}}-(x)^{\frac{3}{2}} \\
&= (3.968)^{\frac{3}{2}}-(4)^{\frac{3}{2}} \\
&= (3.968)^{\frac{3}{2}}-8 \\
\Delta y+8 &= (3.968)^{\frac{3}{2}}
\end{aligned}
$$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x$$

$$= \frac{3}{2}(x)^{\frac{1}{2}}(\Delta x) \qquad \left[\because y = (x)^{\frac{3}{2}}\right]$$

$$= \frac{3}{2}(2)(-0.032) =$$

$$-0.096$$

Hence,

$$(3.968)^{\frac{3}{2}} = 8 + (-0.096)$$

$$= 7.904$$

Thus, the approximate value of $(3.968)^{\frac{3}{2}} = 7.904$.

(xv) $(32.15)^{\frac{1}{5}}$

Consider $y = (x)^{\frac{1}{4}}$

Let $x = 32$ and $\Delta x = 0.15$

Then,

$$\Delta y = (x + \Delta x)^{\frac{1}{4}} - (x)^{\frac{1}{4}}$$

$$= (32.15)^{\frac{1}{5}} - (32)^{\frac{1}{5}}$$

$$= (32.15)^{\frac{1}{5}} - 2$$

$$\Delta y + 2 = (32.15)^{\frac{1}{5}}$$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x$$
$$= \frac{1}{5(x)^{\frac{4}{5}}}(\Delta x) \qquad \left[\because y = (x)^{\frac{1}{5}}\right]$$
$$= \frac{1}{5(2)^4}(0.15)$$
$$= \frac{0.15}{80}$$
$$= 0.00187$$

Hence,

$$(32.15)^{\frac{1}{5}} = 2 + 0.00187$$
$$= 2.00187$$

Thus, the approximate value of $(32.15)^{\frac{1}{5}} = 2.00187$.

Question 2:

Find the approximate value of $f(2.01)$, where $f(x) = 4x^2 + 5x + 2$.

Solution:

Let $x = 2$ and $\Delta x = 0.01$

Then,

$$f(2.01) = f(x+\Delta x)$$
$$= 4(x+\Delta x)^2 + 5(x+\Delta x) + 2$$
$$\Delta y = f(x+\Delta x) - f(x)$$
$$f(x+\Delta x) = f(x) + \Delta y$$
$$\approx f(x) + f'(x)\cdot \Delta x \qquad (\because dx = \Delta x)$$

$$f(2.01) \approx (4x^2 + 5x + 2) + (8x - 5)\Delta x$$
$$= \left[4(2)^2 + 5(2) + 2\right] + \left[8(2) + 5\right](0.01) \qquad [\because x = 2, \Delta x = 0.01]$$
$$= (16 + 10 + 2) + (16 + 5)(0.01)$$
$$= 28 + 21(0.01)$$
$$= 28 + 0.21$$
$$= 28.21$$

Hence, the approximate value of $f(2.01) = 28.21$.

Question 3:

Find the approximate value of $f(5.001)$, where $f(x) = x^3 - 7x^2 + 15$.

Solution:

Let $x = 5$ and $\Delta x = 0.001$

Then,

$$f(5.001) = f(x+\Delta x)$$
$$= (x+\Delta x)^3 - 7(x+\Delta x)^2 + 15$$
$$\Delta y = f(x+\Delta x) - f(x)$$
$$f(x+\Delta x) = f(x) + \Delta y$$
$$\approx f(x) + f'(x)\cdot \Delta x \qquad (\because dx = \Delta x)$$

$$f(5.001) \approx (x^3 - 7x^2 + 15) + (3x^2 - 14x)\Delta x$$
$$= \left[(5)^3 - 7(5)^2 + 15\right] + \left[3(5)^2 - 14(5)\right](0.001) \qquad [\because x = 5, \Delta x = 0.001]$$
$$= (125 - 175 + 15) + (75 - 70)(0.001)$$
$$= (-35) + (5)(0.001)$$
$$= -35 + 0.005$$
$$= -34.995$$

Hence, the approximate value of $f(5.001) = -34.995$.

Question 4:

Find the approximate change in the volume V of a cube of side x meters caused by increasing the side by 1%.

Solution:

The volume of a cube V of side x is given by $V = x^3$

Therefore,

$$\begin{aligned} dV &= \left(\frac{dV}{dx}\right)\Delta x \\ &= \left(3x^2\right)\Delta x \\ &= \left(3x^2\right)(0.01x) \qquad [\because 1\% \text{ of } x \text{ is } 0.01x] \\ &= 0.03x^3 \end{aligned}$$

Hence, the approximate change in the volume of the cube is $0.03x^3 m^3$.

Question 5:

Find the approximate change in the surface area of a cube of side x meters caused by decreasing the side by 1%.

Solution:

The surface area of a cube (S) of side x is given by $S = 6x^2$.

Therefore,

$$\begin{aligned} dS &= \left(\frac{dS}{dx}\right)\Delta x \\ &= (12x)\Delta x \\ &= (12x)(0.01x) \qquad [\because 1\% \text{ of } x \text{ is } 0.01x] \\ &= 0.12x^2 \end{aligned}$$

Hence, the approximate change in the surface area of the cube is $0.12x^2 m^2$.

Question 6:

If the radius of a sphere is measured as $7m$ with an error of $0.02m$, then find the approximate error in calculating its volume.

Solution:

Let r be the radius of the sphere and Δr be the error in measuring the radius.

Then, $r = 7m$ and $\Delta r = 0.02m$

Now, the volume V of the sphere is given by,

$$V = \frac{4}{3}\pi r^3$$

Therefore,

$$\frac{dV}{dr} = 4\pi r^2$$

Hence,

$$\begin{aligned} dV &= \left(\frac{dV}{dr}\right)\Delta r \\ &= \left(4\pi r^2\right)(0.02) \\ &= 4\pi (7)^2 (0.02) \\ &= 3.92\pi \end{aligned}$$

Thus, the approximate error in calculating its volume is $3.92\pi m^3$.

Question 7:

If the radius of a sphere is measured as $9m$ with an error of $0.03m$, then find the approximate error in calculating its surface area.

Solution:

Let r be the radius of the sphere and Δr be the error in measuring the radius.

Then, $r = 9m$ and $\Delta r = 0.03m$

Now, the surface area S of the sphere is given by,

$$S = 4\pi r^2$$

Therefore,

$$\frac{dS}{dr} = 8\pi r$$

Hence,

$$\begin{aligned} dS &= \left(\frac{dS}{dr}\right)\Delta r \\ &= (8\pi r)\Delta r \\ &= 8\pi(9)(0.03) \\ &= 2.16\pi \end{aligned}$$

Thus, the approximate error in calculating its surface area is $2.16\pi m^2$.

Question 8:

If $f(x) = 3x^2 + 15x + 5$, then the approximate value of $f(3.02)$ is

(A) 47.66 **(B)** 57.66 **(C)** 67.66 **(D)** 77.66

Solution:

Let $x = 3$ and $\Delta x = 0.02$

Then,

$$\begin{aligned} f(3.02) = f(x+\Delta x) &= 3(x+\Delta x)^2 + 15(x+\Delta x) + 5 \\ \Delta y &= f(x+\Delta x) - f(x) \\ f(x+\Delta x) &= f(x) + \Delta y \\ &\approx f(x) + f'(x)\cdot\Delta x \qquad (\because dx = \Delta x) \end{aligned}$$

$$\begin{aligned} f(3.02) &\approx (3x^2 + 15x + 5) + (6x + 15)\Delta x \\ &= \left[3(3)^2 + 15(3) + 5\right] + \left[6(3) + 15\right](0.02) \qquad [\because x = 3, \Delta x = 0.02] \\ &= (27 + 45 + 5) + (18 + 15)(0.02) \\ &= 77 + (33)(0.02) \\ &= 77 + 0.66 \\ &= 77.66 \end{aligned}$$

Hence, the approximate value of $f(3.02) = 77.66$.

Thus, the correct option is **D**.

Question 9:

The approximate change in the volume of a cube of side x metres caused by increasing the side by is.

(A) $0.06x^3 m^3$ **(B)** $0.6x^3 m^3$ **(C)** $0.09x^3 m^3$ **(D)** $0.9x^3 m^3$

Solution:

The volume of a cube (V) of side x is given by $V = x^3$.

Therefore,

$$\begin{aligned} dV &= \left(\frac{dV}{dx}\right)\Delta x \\ &= (3x^2)(0.03x) \qquad [\because 3\% \ of \ x \ is \ 0.03x] \\ &= 0.09x^3 \end{aligned}$$

Hence, the approximate change in the volume of the cube $0.09x^3 m^3$.

Thus, the correct option is **C**.

EXERCISE 6.5

Question 1:

Find the maximum and minimum values, if any, of the following functions given by

(i) $f(x)=(2x-1)^2+3$

(ii) $f(x)=9x^2+12x+2$

(iii) $f(x)=-(x-1)^2+10$

(iv) $g(x)=(x)^3+1$

Solution:

(i) The given function is $f(x)=(2x-1)^2+3$

It can be observed that $(2x-1)^2\geq 0$ for every $x\in\mathbf{R}$.

Therefore, $f(x)=(2x-1)^2+3\geq 3$ for every $x\in\mathbf{R}$.

The minimum value of f is attained when $2x-1=0$

$$2x-1=0$$
$$\Rightarrow x=\frac{1}{2}$$

Hence, minimum value of

$$f=f\left(\frac{1}{2}\right)$$
$$=\left(2\left(\frac{1}{2}\right)-1\right)^2+3$$
$$=3$$

Thus, the function f does not have a maximum value.

(ii) The given function is $f(x)=9x^2+12x+2$

It can be observed that $(3x+2)^2\geq 0$ for every $x\in\mathbf{R}$.

Therefore, $f(x)=(3x+2)^2-2\geq -2$ for every $x\in\mathbf{R}$.

The minimum value of f is attained when $3x+2=0$

$$3x+2=0$$
$$\Rightarrow x=\frac{-2}{3}$$

Therefore, Minimum value of

$$f = f\left(-\frac{2}{3}\right)$$
$$= \left(3\left(-\frac{2}{3}\right)+2\right)^2 - 2$$
$$= -2$$

Hence, the function f does not have a maximum value.

(iii) The given function is $f(x) = -(x-1)^2 + 10$

It can be observed that $(x-1)^2 \geq 0$ for every $x \in \mathbf{R}$.

Therefore, $f(x) = -(x-1)^2 + 10 \leq 10$ for every $x \in \mathbf{R}$.

The maximum value of f is attained when $(x-1) = 0$

$$(x-1) = 0$$
$$\Rightarrow x = 1$$

Therefore, Maximum value of

$$f = f(1)$$
$$= -(1-1)^2 + 10$$
$$= 10$$

Hence, the function f does not have a minimum value.

(iv) The given function is $g(x) = (x)^3 + 1$

Hence, function g neither has a maximum value nor a minimum value.

Question 2:

Find the maximum and minimum values, if any, of the following functions given by:

(i) $f(x) = |x+2| - 1$

(ii) $g(x) = -|x+1| + 3$

(iii) $h(x) = \sin(2x) + 5$

(iv) $f(x) = |\sin 4x + 3|$

(v) $h(x) = x + 1, x \in (-1, 1)$

Solution:

(i) The given function is $f(x) = |x+2| - 1$

It can be observed that $|x+2| \geq 0$ for every $x \in \mathbf{R}$.

Therefore, $f(x) = |x+2| - 1 \geq -1$ for every $x \in \mathbf{R}$.

The minimum value of *f* is attained when $|x+2| = 0$

$$|x+2| = 0$$
$$\Rightarrow x = -2$$

Therefore, Minimum value of

$$f = f(-2)$$
$$= |-2+1| - 1$$
$$= -1$$

Hence, the function f does not have a maximum value.

(ii) The given function is $g(x) = -|x+1| + 3$

It can be observed that $-|x+1| \leq 0$ for every $x \in \mathbf{R}$.

Therefore, $f(x) = -|x+1| + 3 \leq 3$ for every $x \in \mathbf{R}$.

The maximum value of *g* is attained when $|x+1| = 0$

$$|x+1| = 0$$
$$\Rightarrow x = -1$$

Therefore, maximum value of

$$g = g(-1)$$
$$= -|-1+1| + 3$$
$$= 3$$

Hence, the function g does not have a minimum value.

(iii) The given function is $h(x) = \sin(2x) + 5$

We know that $-1 \leq \sin 2x \leq 1$

Therefore,

$$\Rightarrow -1+5 \leq \sin 2x \leq 1+5$$
$$\Rightarrow 4 \leq \sin 2x + 5 \leq 6$$

Hence, the maximum and minimum values of h are 6 and 4, respectively.

(iv) The given function is $f(x) = |\sin 4x + 3|$

We know that $-1 \leq \sin 4x \leq 1$

Therefore,

$\Rightarrow 2 \le \sin 4x + 3 \le 4$

$\Rightarrow 2 \le |\sin 4x + 3| \le 4$

Hence, the maximum and minimum values of f are 4 and 2, respectively.

(v) The given function is $h(x) = x + 1, x \in (-1, 1)$

Here, if a point x_0 is closest to -1, then we find $\frac{x_0}{2} + 1 > x_0 + 1$ for all $x_0 \in (-1, 1)$.

Also, if x_1 is closest to 1, then find $x_1 + 1 < \frac{x_1 + 1}{2} + 1$ for all $x_1 \in (-1, 1)$.

Hence, function $h(x)$ has neither maximum nor minimum value in $(-1, 1)$.

Question 3:

Find the local maxima and minima, if any, of the following functions. Find also the local maximum and the local minimum values, as the case may be:

(i) $f(x) = x^2$

(ii) $g(x) = x^3 - 3x$

(iii) $h(x) = \sin x + \cos x, 0 < x < \frac{\pi}{2}$

(iv) $f(x) = \sin x - \cos x, 0 < x < 2\pi$

(v) $f(x) = x^3 - 6x^2 + 9x + 15$

(vi) $g(x) = \frac{x}{2} + \frac{2}{x}, x > 0$

(vii) $g(x) = \frac{1}{x^2 + 2}$

(viii) $f(x) = x\sqrt{1 - x}, 0 < x < 1$

Solution:

(i) $f(x) = x^2$

Therefore,

$$f'(x) = 2x$$

Now,

$$f'(x) = 0$$

$$\Rightarrow x = 0$$

Thus, $x = 0$ is the only critical point which could possibly be the point of local maxima or local minima of f.

We have $f''(0)=2$, which is positive.

Therefore, by second derivative test, $x=0$ is a point of local minima and local minimum value of f at $x=0$ is $f(0)=0$.

(ii) $g(x)=x^3-3x$

Therefore,

$$g'(x)=3x^2-3$$

Now,

$$g'(x)=0$$
$$\Rightarrow 3x^2-3=0$$
$$\Rightarrow x=\pm 1$$

Also,

$$g''(x)=6x$$
$$g''(1)=6>0$$
$$g''(-1)=-6<0$$

By second derivative test, $x=1$ is a point of local minima and local minimum value of g at $x=1$ is

$$g(1)=1^3-3$$
$$=1-3$$
$$=-2$$

However, $x=-1$ is a point of local maxima and local maximum value of g at $x=-1$ is

$$g(-1)=(-1)^3-3(-1)$$
$$=-1+3$$
$$=2$$

(iii) $h(x)=\sin x+\cos x, 0<x<\frac{\pi}{2}$

Therefore,

$$h'(x)=\cos x-\sin x$$

Now,

$$h'(x) = 0$$
$$\Rightarrow \cos x - \sin x = 0$$
$$\Rightarrow \sin x = \cos x$$
$$\Rightarrow \tan x = 1$$
$$\Rightarrow x = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$$

Also,

$$h''(x) = -\sin x - \cos x$$
$$= -(\sin x + \cos x)$$

Hence,

$$h''\left(\frac{\pi}{4}\right) = -\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$$
$$= \frac{-2}{\sqrt{2}}$$
$$= -\sqrt{2} < 0$$

Therefore, by second derivative test, $x = \frac{\pi}{4}$ is a point of local maxima and the local maximum value of h at $x = \frac{\pi}{4}$ is

$$h\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4}$$
$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$
$$= \sqrt{2}$$

(iv) $f(x) = \sin x - \cos x, 0 < x < 2\pi$

Therefore,

$$f'(x) = \cos x + \sin x$$

Now,

$$f'(x) = 0$$
$$\Rightarrow \cos x + \sin x = 0$$
$$\Rightarrow \sin x = -\cos x$$
$$\Rightarrow \tan x = -1$$
$$\Rightarrow x = \frac{3\pi}{4}, \frac{7\pi}{4} \in (0, 2\pi)$$

Also,

$$f''(x) = -\sin x + \cos x$$

Hence,

$$f''\left(\frac{3\pi}{4}\right) = -\sin\left(\frac{3\pi}{4}\right) + \cos\left(\frac{3\pi}{4}\right)$$
$$= \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)$$
$$= -\sqrt{2} < 0$$
$$f''\left(\frac{7\pi}{4}\right) = -\sin\left(\frac{7\pi}{4}\right) + \cos\left(\frac{7\pi}{4}\right)$$
$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$$
$$= \sqrt{2} > 0$$

Therefore, by second derivative test, $x = \left(\frac{3\pi}{4}\right)$ is a point of local maxima and the local maximum value of f at $x = \left(\frac{3\pi}{4}\right)$ is

$$f\left(\frac{3\pi}{4}\right) = \sin\left(\frac{3\pi}{4}\right) - \cos\left(\frac{3\pi}{4}\right)$$
$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$$
$$= \sqrt{2}$$

However, $x = \left(\frac{7\pi}{4}\right)$ is a point of local minima and the local minimum value of f at $x = \left(\frac{7\pi}{4}\right)$ is

$$f\left(\frac{7\pi}{4}\right) = \sin\left(\frac{7\pi}{4}\right) - \cos\left(\frac{7\pi}{4}\right)$$
$$= \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)$$
$$= -\sqrt{2}$$

(v) $f(x) = x^3 - 6x^2 + 9x + 15$

Therefore,

$$f'(x) = 3x^2 - 12x + 9$$

Now,

$$f'(x) = 0$$
$$\Rightarrow 3x^2 - 12x + 9 = 0$$
$$\Rightarrow 3(x-1)(x-3) = 0$$
$$\Rightarrow x = 1, 3$$

Also,

$$f''(x) = 6x - 12$$
$$= 6(x-2)$$

Hence,

$$f''(1) = 6(1-2) = -6 < 0$$
$$f''(3) = 6(3-2) = 6 > 0$$

Therefore, by second derivative test, $x = 1$ is a point of local maxima and the local maximum value of f at $x = 1$ is

$$f(1) = 1 - 6 + 9 + 15$$
$$= 19$$

However, $x = 3$ is a point of local minima and the local minimum value of f at $x = 3$ is

$$f(3) = 27 - 54 + 27 + 15$$
$$= 15$$

(vi) $g(x) = \frac{x}{2} + \frac{2}{x}, x > 0$

Therefore,

$$g'(x) = \frac{1}{2} - \frac{2}{x^2}$$

Now,

$$g'(x) = 0$$
$$\Rightarrow \frac{1}{2} - \frac{2}{x^2} = 0$$
$$\Rightarrow \frac{2}{x^2} = \frac{1}{2}$$
$$\Rightarrow x^2 = 4$$
$$\Rightarrow x = \pm 2$$

Since, $x > 0$, we take $x = 2$

Hence,

$$g''(x) = \frac{4}{x^3}$$
$$g''(2) = \frac{4}{2^3} = \frac{1}{2} > 0$$

Therefore, by second derivative test, $x = 2$ is a point of local minima and the local minimum value of g at $x = 2$ is

$$\begin{aligned} g(2) &= \frac{2}{2} + \frac{2}{2} \\ &= 1 + 1 \\ &= 2 \end{aligned}$$

(vii) $g(x) = \frac{1}{x^2 + 2}$

Therefore,

$$g'(x) = \frac{-(2x)}{(x^2 + 2)^2}$$

Now,

$$\begin{aligned} & g'(x) = 0 \\ & \Rightarrow \frac{-(2x)}{(x^2 + 2)^2} = 0 \\ & \Rightarrow x = 0 \end{aligned}$$

Now, for values close to $x = 0$ and to the left of 0, $g'(x) > 0$.

Also, for values close to $x = 0$ and to the right of 0, $g'(x) < 0$.
Therefore, by first derivative test, $x = 0$ is a point of local maxima and the local maximum value of

$$\begin{aligned} g(0) &= \frac{1}{0 + 2} \\ &= \frac{1}{2} \end{aligned}$$

(viii) $f(x) = x\sqrt{1 - x}, 0 < x < 1$

Therefore,

$$f'(x)=\sqrt{1-x}+x\left(\frac{1}{2\sqrt{1-x}}\right)(-1)$$

$$=\sqrt{1-x}-\frac{x}{2\sqrt{1-x}}$$

$$=\frac{2(1-x)-x}{2\sqrt{1-x}}$$

$$=\frac{2-3x}{2\sqrt{1-x}}$$

Now,

$$f'(x)=0$$

$$\Rightarrow \frac{2-3x}{2\sqrt{1-x}}=0$$

$$\Rightarrow 2-3x=0$$

$$\Rightarrow x=\frac{2}{3}$$

Also,

$$f''(x)=\frac{1}{2}\left[\frac{\sqrt{1-x}(-3)-(2-3x)\left(\frac{-1}{2\sqrt{1-x}}\right)}{1-x}\right]$$

$$=\frac{\sqrt{1-x}(-3)+(2-3x)\left(\frac{1}{2\sqrt{1-x}}\right)}{2(1-x)}$$

$$=\frac{-6(1-x)+(2-3x)}{4(1-x)\left(\sqrt{1-x}\right)}$$

$$=\frac{3x-4}{4(1-x)^{\frac{3}{2}}}$$

Hence,

$$f''\left(\frac{2}{3}\right)=\frac{3\left(\frac{2}{3}\right)-4}{4\left(1-\frac{2}{3}\right)^{\frac{3}{2}}}$$

$$=\frac{2-4}{4\left(\frac{1}{3}\right)^{\frac{3}{2}}}$$

$$=\frac{-1}{2\left(\frac{1}{3}\right)^{\frac{3}{2}}}<0$$

Therefore, by second derivative test, $x=\frac{2}{3}$ is a point of local maxima and the local maximum value of f at $x=\frac{2}{3}$ is

$$f\left(\frac{2}{3}\right)=\frac{2}{3}\sqrt{1-\frac{2}{3}}$$

$$=\frac{2}{3}\sqrt{\frac{1}{3}}$$

$$=\frac{2}{3\sqrt{3}}$$

$$=\frac{2\sqrt{3}}{9}$$

Question 4:

Prove that the following functions do not have maxima or minima:

(i) $f(x)=e^x$

(ii) $g(x)=\log x$

(iii) $h(x)=x^3+x^2+x+1$

Solution:

(i) $f(x)=e^x$

Therefore,

$$f'(x)=e^x$$

Now,

$$f'(x) = 0$$
$$\Rightarrow e^x = 0$$

But the exponential function can never assume 0 for any value of x .

Therefore, there does not exist $c \in \mathbf{R}$ such that $f'(c) = 0$

Hence, function f does not have maxima or minima.

(ii) $g(x) = \log x$

Therefore,

$$g'(x) = \frac{1}{x}$$

Since $\log x$ is defined for a positive number $x, g'(x) > 0$ for any x.

Therefore, there does not exist $c \in \mathbf{R}$ such that $g'(c) = 0$.

Hence, function g does not have maxima or minima.

(iii) $h(x) = x^3 + x^2 + x + 1$

Therefore,

$$h'(x) = 3x^2 + 2x + 1$$

Now,

$$h'(x) = 0$$
$$\Rightarrow 3x^2 + 2x + 1 = 0$$
$$\Rightarrow x = \frac{-2 \pm 2\sqrt{2}i}{6}$$
$$\Rightarrow x = \frac{-1 \pm \sqrt{2}i}{3} \neq \mathbf{R}$$

Therefore, there does not exist $c \in \mathbf{R}$ such that $h'(c) = 0$.

Hence, function h does not have maxima or minima.

Question 5:

Find the absolute maximum value and the absolute minimum value of the following functions in the given intervals:

(i) $f(x)=x^3, x\in[-2,2]$

(ii) $f(x)=\sin x+\cos x, x\in[0,\pi]$

(iii) $f(x)=4x-\frac{1}{2}x^2, x\in\left[-2,\frac{9}{2}\right]$

(iv) $f(x)=(x-1)^2+3, x\in[-3,1]$

Solution:

(i) The given function is $f(x)=x^3$

Therefore,

$$f'(x)=3x^2$$

Now,

$$f'(x)=0$$
$$\Rightarrow 3x^2=0$$

Then, we evaluate the value of f at critical point $x=0$ and at end points of the interval $[-2,2]$.

Therefore,

$$f(0)=0$$
$$f(-2)=(-2)^3$$
$$=-8$$
$$f(2)=(2)^3$$
$$=8$$

Hence, we can conclude that the absolute maximum value of f on $[-2,2]$ is 8 occurring at $x=2$.

Also, the absolute minimum value of f on $[-2,2]$ is -8 occurring at $x=-2$.

(ii) The given function is $f(x)=\sin x+\cos x$

Therefore,

$$f'(x)=\cos x-\sin x$$

Now,

$$f'(x) = 0$$
$$\Rightarrow \cos x - \sin x = 0$$
$$\Rightarrow \sin x = \cos x$$
$$\Rightarrow \tan x = 1$$
$$\Rightarrow x = \frac{\pi}{4}$$

Then, we evaluate the value of f at critical point $x = \frac{\pi}{4}$ and at the end points of the interval $[0, \pi]$.

Therefore,

$$f\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right)$$
$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}}$$
$$= \sqrt{2}$$
$$f(0) = \sin 0 + \cos 0$$
$$= 0 + 1$$
$$= 1$$
$$f(\pi) = \sin \pi + \cos \pi$$
$$= 0 - 1$$
$$= -1$$

Hence, we can conclude that the absolute maximum value of f on $[0, \pi]$ is $\sqrt{2}$ occurring at $x = \frac{\pi}{4}$.

Also, the absolute minimum value of f on $[0, \pi]$ is -1 occurring at $x = \pi$.

(iii) The given function is $f(x) = 4x - \frac{1}{2}x^2$

Therefore,

$$f'(x) = 4 - \frac{1}{2}(2x)$$
$$= 4 - x$$

Now,

$$f'(x) = 0$$
$$\Rightarrow 4 - x = 0$$
$$\Rightarrow x = 4$$

Then, we evaluate the value of f at critical point $x = 4$ and at end points of the interval $\left[-2, \frac{9}{2}\right]$.

Therefore,

$$f(4) = 16 - \frac{1}{2}(16)$$
$$= 16 - 8$$
$$= 8$$
$$f(-2) = -8 - \frac{1}{2}(4)$$
$$= -8 - 2$$
$$= -10$$
$$f\left(\frac{9}{2}\right) = 18 - \frac{1}{2}\left(\frac{9}{2}\right)^2$$
$$= 18 - \frac{81}{8}$$
$$= 18 - 10.125$$
$$= 7.875$$

Hence, we can conclude that the absolute maximum value of f on $\left[-2, \frac{9}{2}\right]$ is 8 occurring at $x = 4$.

Also, the absolute minimum value of f on $\left[-2, \frac{9}{2}\right]$ is -10 occurring at $x = -2$.

(iv) The given function is $f(x) = (x-1)^2 + 3$

Therefore,

$$f'(x) = 2(x-1)$$

Now,

$$f'(x) = 0$$
$$\Rightarrow 2(x-1) = 0$$
$$\Rightarrow x = 1$$

Then, we evaluate the value of f at critical point $x=1$ and at end points of the interval $[-3,1]$.

$$\begin{aligned} f(1) &= (1-1)^3+3 \\ &= 3 \\ f(-3) &= (-3-1)^2+3 \\ &= 16+3 \\ &= 19 \end{aligned}$$

Hence, we can conclude that the absolute maximum value of f on $[-3,1]$ is 19 occurring at $x=-3$.

Also, the absolute minimum value of f on $[-3,1]$ is 3 occurring at $x=1$.

Question 6:

Find the absolute maximum profit that a company can make, if the profit function is given by: $p(x)=41-24x-18x^2$

Solution:

The profit function is given as $p(x)=41-24x-18x^2$

Therefore,

$$p'(x)=-24-36x$$

Now,

$$\begin{aligned} & p'(x)=0 \\ & \Rightarrow 24-36x=0 \\ & \Rightarrow x=-\frac{24}{36} \\ & \Rightarrow x=\frac{-2}{3} \end{aligned}$$

Also,

$$p''\left(\frac{-2}{3}\right)=-36<0$$

By second derivative test, $x=-\frac{2}{3}$ is the point of local maxima of p.

Therefore, maximum profit

$$p\left(-\frac{2}{3}\right)=41-24\left(-\frac{2}{3}\right)-18\left(-\frac{2}{3}\right)^2$$
$$=41+16-8$$
$$=49$$

Hence, the maximum profit that the company can make is 49 units.

Question 7:

Find both the maximum value and the minimum value of $3x^4-8x^3+12x^2-48x+25$ on the interval $[0,3]$.

Solution:

Let $f(x)=3x^4-8x^3+12x^2-48x+25$

Therefore,

$$f'(x)=12x^3-24x^2+24x-48$$
$$=12\left(x^3-2x^2+2x-4\right)$$
$$=12\left[x^2(x-2)+2(x-2)\right]$$
$$=12(x-2)\left(x^2+2\right)$$

Now,

$$f'(x)=0$$
$$\Rightarrow x-2=0$$
$$\Rightarrow x=2$$

Now, we evaluate the value of f at critical point $x=2$ and at the end points of the interval $[0,3]$.

Therefore,

$$f(2)=3(2)^4-8(2)^3+12(2)^2-48(2)+25$$
$$=48-64+48-96+25$$
$$=-39$$
$$f(0)=3(0)^4-8(0)^3+12(0)^2-48(0)+25$$
$$=25$$
$$f(3)=3(3)^4-8(3)^3+12(3)^2-48(3)+25$$
$$=243-216+108-144+25$$
$$=16$$

Hence, we can conclude that the absolute maximum value of f on $[0,3]$ is 25 occurring at $x=0$ and the absolute minimum value of f at $[0,3]$ is -39 occurring at $x=2$.

Question 8:

At what points in the interval $[0,2\pi]$ does the function $\sin 2x$ attain its maximum value?

Solution:

Let $f(x)=\sin 2x$

Therefore,

$$f'(x)=2\cos 2x$$

Now,

$$\begin{aligned}
&f'(x)=0\\
&\Rightarrow 2\cos 2x=0\\
&\Rightarrow 2x=\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2},\frac{7\pi}{2}\\
&\Rightarrow x=\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}
\end{aligned}$$

Now, we evaluate the value of f at critical point $x=\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}$ and at the end points of the interval $[0,2\pi]$.

Therefore,

$$\begin{aligned}
&f\left(\frac{\pi}{4}\right)=\sin\frac{\pi}{2}=1\\
&f\left(\frac{3\pi}{4}\right)=\sin\frac{3\pi}{2}=-1\\
&f\left(\frac{5\pi}{4}\right)=\sin\frac{5\pi}{2}=1\\
&f\left(\frac{7\pi}{4}\right)=\sin\frac{7\pi}{2}=-1\\
&f(0)=\sin 0=0\\
&f(2\pi)=\sin 2\pi=0
\end{aligned}$$

Hence, we can conclude that the absolute maximum value of f on $[0, 2\pi]$ is occurring at $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$.

Question 9:

What is the maximum value of the function $\sin x + \cos x$?

Solution:

Let $f(x) = \sin x + \cos x$

Therefore,

$$f'(x) = \cos x - \sin x$$

Now,

$$f'(x) = 0$$
$$\Rightarrow \cos x - \sin x = 0$$
$$\Rightarrow \sin x = \cos x$$
$$\Rightarrow \tan x = 1$$
$$\Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4} \ldots$$

Hence,

$$\begin{aligned} f''(x) &= -\sin x - \cos x \\ &= -(\sin x + \cos x) \end{aligned}$$

Now, $f''(x)$ will be negative when $(\sin x + \cos x)$ is positive i.e., when $\sin x$ and $\cos x$ are both positive.

Also, we know that $\sin x$ and $\cos x$ both are positive in the first quadrant.

Then, $f''(x)$ will be negative when $x \in \left(0, \frac{\pi}{2}\right)$.

Thus, we consider $x = \frac{\pi}{4}$

$$\begin{aligned} f''\left(\frac{\pi}{4}\right) &= -\left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right) \\ &= -\left(\frac{2}{\sqrt{2}}\right) \\ &= -\sqrt{2} < 0 \end{aligned}$$

By second derivative test, f will be the maximum at $x = \frac{\pi}{4}$ and the maximum value of f is

$$\begin{aligned} f\left(\frac{\pi}{4}\right) &= \sin\frac{\pi}{4} + \cos\frac{\pi}{4} \\ &= \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \\ &= \frac{2}{\sqrt{2}} \\ &= \sqrt{2} \end{aligned}$$

Question 10:

Find the maximum value of $2x^3 - 24x + 107$ in the interval $[1, 3]$. Find the maximum value of the same function in $[-3, -1]$.

Solution:

Let $f(x) = 2x^3 - 24x + 107$

Therefore,

$$\begin{aligned} f'(x) &= 6x^2 - 24 \\ &= 6(x^2 - 4) \end{aligned}$$

Now,

$$\begin{aligned} f'(x) &= 0 \\ &\Rightarrow 6(x^2 - 4) = 0 \\ &\Rightarrow x^2 = 4 \\ &\Rightarrow x = \pm 2 \end{aligned}$$

We first consider the interval $[1,3]$.

Then, we evaluate the value of f at the critical point $x = 2 \in [1,3]$ and at the end points of the interval $[1,3]$.

Hence,

$$\begin{aligned} f(2) &= 2(2)^3 - 24(2) + 107 \\ &= 16 - 48 + 107 \\ &= 75 \end{aligned}$$

$$\begin{aligned} f(1) &= 2(1)^3 - 24(1) + 107 \\ &= 2 - 24 + 107 \\ &= 85 \end{aligned}$$

$$\begin{aligned} f(3) &= 2(3)^3 - 24(3) + 107 \\ &= 54 - 72 + 107 \\ &= 89 \end{aligned}$$

Thus, the absolute maximum value of $f(x)$ in the interval $[1,3]$ is 89 occurring at $x = 3$.

Next, we first consider the interval $[-3,-1]$.

Then, we evaluate the value of f at the critical point $x = -2 \in [-3,-1]$ and at the end points of the interval $[-3,-1]$.

Hence,

$$f(-3) = 2(-3)^3 - 24(-3) + 107$$
$$= -54 + 72 + 107$$
$$= 125$$
$$f(-1) = 2(-1)^3 - 24(-1) + 107$$
$$= -2 + 24 + 107$$
$$= 129$$
$$f(-2) = 2(-2)^3 - 24(-2) + 107$$
$$= -16 + 48 + 107$$
$$= 139$$

Hence, the absolute maximum value of $f(x)$ in the interval $[-3, -1]$ is 139 occurring at $x = -2$.

Question 11:

It is given that at $x = 1$, the function $x^4 - 62x^2 + ax + 9$ attains its maximum value, on the interval $[0, 2]$. Find the value of a.

Solution:

Let $f(x) = x^4 - 62x^2 + ax + 9$

Therefore,

$$f'(x) = 4x^3 - 124x + a$$

It is given that function f attains its maximum value on the interval $[0, 2]$ at $x = 1$.
Hence,

$$f'(1) = 0$$
$$\Rightarrow 4x^3 - 124x + a = 0$$
$$\Rightarrow 4 - 124 + a = 0$$
$$\Rightarrow -120 + a = 0$$
$$\Rightarrow a = 120$$

Thus, the value of $a = 120$.

Question 12:

Find the maximum and minimum values of $x + \sin 2x$ on $[0, 2\pi]$.

Solution:

Let $f(x)=x+\sin 2x$

Therefore,

$$f'(x)=1+2\cos 2x$$

Now,

$$f'(x)=0$$
$$\Rightarrow 1+2\cos 2x=0$$
$$\Rightarrow \cos 2x=\frac{-1}{2}=-\cos\frac{\pi}{3}=\cos\left(\pi-\frac{\pi}{3}\right)=\cos\frac{2\pi}{3}$$
$$\Rightarrow 2x=2n\pi\pm\frac{2\pi}{3} \qquad [n\in\mathbf{Z}]$$
$$\Rightarrow x=n\pi\pm\frac{\pi}{3} \qquad [n\in\mathbf{Z}]$$
$$\Rightarrow x=\frac{\pi}{3},\frac{2\pi}{3},\frac{4\pi}{3},\frac{5\pi}{3}\in[0,2\pi]$$

Then, we evaluate the value of f at critical points $x=\frac{\pi}{3},\frac{2\pi}{3},\frac{4\pi}{3},\frac{5\pi}{3}$ and at the end points of the interval $[0,2\pi]$.

Hence,

$$f\left(\frac{\pi}{3}\right)=\left(\frac{\pi}{3}\right)+\sin 2\left(\frac{\pi}{3}\right)$$
$$=\frac{\pi}{3}+\frac{\sqrt{3}}{2}$$
$$f\left(\frac{2\pi}{3}\right)=\left(\frac{2\pi}{3}\right)+\sin 2\left(\frac{2\pi}{3}\right)$$
$$=\frac{2\pi}{3}-\frac{\sqrt{3}}{2}$$
$$f\left(\frac{4\pi}{3}\right)=\left(\frac{4\pi}{3}\right)+\sin 2\left(\frac{4\pi}{3}\right)$$
$$=\frac{4\pi}{3}+\frac{\sqrt{3}}{2}$$
$$f\left(\frac{5\pi}{3}\right)=\left(\frac{5\pi}{3}\right)+\sin 2\left(\frac{5\pi}{3}\right)$$
$$=\frac{5\pi}{3}-\frac{\sqrt{3}}{2}$$

$$
\begin{aligned}
f(0) &= 0+\sin 0 \\
&= 0 \\
f(2\pi) &= 2\pi+\sin 4\pi \\
&= 2\pi+0 \\
&= 2\pi
\end{aligned}
$$

Hence, we can conclude that the absolute maximum value of $f(x)$ in the interval $[0,2\pi]$ is 2π occurring at $x=2\pi$ and the absolute minimum value of $f(x)$ in the interval $[0,2\pi]$ is 0 occurring at $x=0$.

Question 13:

Find two numbers whose sum is 24 and whose product is as large as possible.

Solution:

Let one number be x.

Then, the other number be $(24-x)$.

Let $P(x)$ denote the product of the two numbers.

Thus, we have:

$$
\begin{aligned}
P(x) &= x(24-x) \\
&= 24x-x^2
\end{aligned}
$$

Therefore,

$$P'(x) = 24 - 2x$$
$$P''(x) = -2$$

Now,

$$P'(x) = 0$$
$$\Rightarrow 24 - 2x = 0$$
$$\Rightarrow 24 = 2x$$
$$\Rightarrow x = 12$$

Also,

$$P''(12) = -2 < 0$$

By second derivative test, $x = 12$ is the point of local maxima of P .

Thus, the numbers are 12 and $(24-12) = 12$.

Hence, the product of the numbers is the maximum when the numbers are 12 each.

Question 14:

Find two positive numbers x and y such that $x + y = 60$ and xy^3 is maximum.

Solution:

The two numbers are x and y such that $x + y = 60$
Therefore,

$$\Rightarrow y = 60 - x$$

Let, $f(x) = xy^3$

$$f(x) = x(60-x)^3$$

Therefore,

$$\begin{aligned} f'(x) &= (60-x)^3 - 3x(60-x)^2 \\ &= (60-x)^2[60 - x - 3x] \\ &= (60-x)^2(60-4x) \end{aligned}$$

$$\begin{aligned} f''(x) &= -2(60-x)(60-4x) - 4(60-x)^2 \\ &= -2(60-x)[60 - 4x + 2(60-x)] \\ &= -2(60-x)(180-6x) \\ &= -12(60-x)(30-x) \end{aligned}$$

Now,

$$f'(x) = 0$$
$$\Rightarrow x = 60 \text{ or } x = 15$$

When, $x = 60$
Then,

$$f''(x) = 0$$

When, $x = 15$
Then,

$$f''(x) = -12(60-15)(30-15)$$
$$= -12 \times 45 \times 15 < 0$$

By second derivative test, $x = 15$ is a point of local maxima of f.

Thus, function xy^3 is maximum when $x = 15$ and $y = 60 - 15 = 45$.

Hence, the required numbers are 15 and 45.

Question 15:

Find two positive numbers x and y such that their sum is 35 and the product x^2y^5 is a maximum.

Solution:

Let one number be x.

Then, the other number is $y = (35 - x)$.

Let $P(x) = x^2y^5$

Then we have, $P(x) = x^2(35 - x)^5$

Therefore,

$$\begin{aligned}P'(x) &= 2x(35-x)^5 - 5x^2(35-x)^4 \\ &= x(35-x)^4\left[2(35-x)-5x\right] \\ &= x(35-x)^4(70-7x) \\ &= 7x(35-x)^4(10-x)\end{aligned}$$

$$\begin{aligned}P''(x) &= 7(35-x)^4(10-x)+7x\left[-(35-x)^4-4(35-x)^3(10-x)\right] \\ &= 7(35-x)^4(10-x)-7x(35-x)^4-28x(35-x)^3(10-x) \\ &= 7(35-x)^3\left[(35-x)(10-x)-x(35-x)-4x(10-x)\right] \\ &= 7(35-x)^3\left[350-45x+x^2-35x+x^2-40x+4x^2\right] \\ &= 7(35-x)^3\left(6x^2-120x+350\right)\end{aligned}$$

Now,

$$\begin{aligned}&P'(x)=0 \\ &\Rightarrow x=0, x=35, x=10\end{aligned}$$

When, $x=35$
Then,

$$\begin{aligned}&P'(x)=P(x)=0 \\ &\Rightarrow y=35-35=0\end{aligned}$$

This will make the product x^2y^5 equal to 0.

When, $x=0$
Then,

$$\Rightarrow y=35-0=35$$

This will make the product x^2y^5 equal to 0.

Hence, $x=0$ and $x=35$ cannot be the possible values of x.

When, $x=10$
Then,

$$\begin{aligned}P''(x) &= 7(35-10)^3(6\times100-120\times10+350) \\ &= 7(25)^3(-250)<0\end{aligned}$$

By second derivative test, $P(x)$ will be the maximum when $x=10$ and $y=35-10=25$.

Hence, the required numbers are 10 and 25.

Question 16:
Find two positive numbers whose sum is 16 and the sum of whose cubes is minimum.

Solution:

Let one number be x.

Then, the other number be $(16-x)$.

Let the sum of the cubes of these numbers be denoted by $S(x)$.

Then,

$$S(x)=x^3+(16-x)^3$$

Therefore,

$$S'(x)=3x^2-3(16-x)^2$$
$$S''(x)=6x+6(16-x)$$

Now,

$$S'(x)=0$$
$$\Rightarrow 3x^2-3(16-x)^2=0$$
$$\Rightarrow x^2-(16-x)^2=0$$
$$\Rightarrow x^2-256-x^2+32x=0$$
$$\Rightarrow x=\frac{256}{32}$$
$$\Rightarrow x=8$$

Also,

$$S''(8)=6(8)+6(16-8)$$
$$=48+48$$
$$=96>0$$

By second derivative test, $x=8$ is the point of local minima of S.

Thus, the numbers are 8 and $(16-8)=8$.

Hence, the sum of the cubes of the numbers is the minimum when the numbers are 8 each.

Question 17:

A square piece of tin of side $18cm$ is to be made into a box without top, by cutting a square from each corner and folding up the flaps to form the box. What should be the side of the square to be cut off so that the volume of the box is the maximum possible?

Solution:

Let the side of the square to be cut off be $x\ cm$.

Then, the length and the breadth of the box will be $(18-2x)cm$ each and the height of the box be $x\ cm$.

Therefore, the volume $V(x)$ of the box is given by,

$$V(x)=x(18-2x)^2$$

Hence,

$$\begin{aligned} V'(x) &= (18-2x)^2-4x(18-2x) \\ &= (18-2x)[18-2x-4x] \\ &= (18-2x)(18-6x) \\ &= 6\times 2(9-x)(3-x) \\ &= 12(9-x)(3-x) \\ V''(x) &= 12(-9-x)(3-x) \\ &= -12(9-x+3-x) \\ &= -12(12-2x) \\ &= -24(6-x) \end{aligned}$$

Now,

$$\begin{aligned} &V'(x)=0 \\ &\Rightarrow x=9, x=3 \end{aligned}$$

If, $x=9$ then the length and the breadth will become 0.
Hence, $x\neq 9$

When, $x = 3$
Then,

$$V''(3) = -24(6-3)$$
$$= -72 < 0$$

By second derivative test, $x = 3$ is the point of local maxima of V.

Hence, if we remove a square of side $3cm$ from each corner of the square tin and make a box from the remaining sheet, then the volume of the box obtained is the largest possible.

Question 18:
A rectangular sheet of tin $45cm$ by $24cm$ is to be made into a box without top, by cutting off square from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is the maximum possible?

Solution:
Let the side of the square to be cut off be $x\ cm$.
Then, the height of the box is $x\ cm$, the length is $(45-2x)cm$ and the breadth is $(24-2x)cm$.

Therefore, the volume $V(x)$ of the box is given by,

$$V(x) = x(45-2x)(24-2x)$$
$$= x(1080 - 90x - 48x + 4x^2)$$
$$= 4x^3 - 138x^2 + 1080x$$

Hence,

$$V'(x) = 12x^2 - 276x + 1080$$
$$= 12(x^2 - 23x + 90)$$
$$= 12(x-18)(x-5)$$
$$V''(x) = 24x - 276$$
$$= 12(2x - 23)$$

Now,

$$V'(x) = 0$$
$$\Rightarrow x = 18, x = 5$$

It is not possible to cut off a square of side $18cm$ from each corner of the rectangular sheet. Thus, x cannot be equal to 18.

When, $x=5$

Then,

$$\begin{aligned} V''(5) &= 12(2(5)-23) \\ &= 12(10-23) \\ &= 12(-13) \\ &= -156 < 0 \end{aligned}$$

By second derivative test, $x=5$ is the point of local maxima.

Hence, the side of the square to be cut off to make the volume of the box maximum possible is $5cm$.

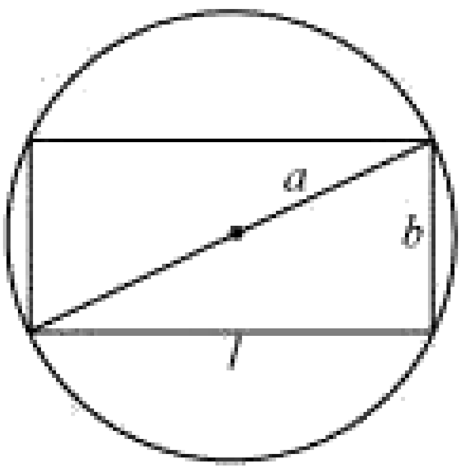
Question 19:

Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area.

Solution:

Let a rectangle of length l and breadth b be inscribed in the given circle of radius a.

Then, the diagonal passes through the centre and is of length $2a\ cm$.



Now, by applying the Pythagoras theorem, we have:

$$(2a)^2 = l^2 + b^2$$
$$\Rightarrow b^2 = 4a^2 - l^2$$
$$\Rightarrow b = \sqrt{4a^2 - l^2}$$

Area of triangle, $A = l\sqrt{4a^2 - l^2}$

Therefore,

$$\frac{dA}{dl} = \sqrt{4a^2 - l^2} + l\frac{1}{2\sqrt{4a^2 - l^2}}(-2l)$$
$$= \sqrt{4a^2 - l^2} - \frac{l^2}{\sqrt{4a^2 - l^2}}$$
$$= \frac{4a^2 - 2l^2}{\sqrt{4a^2 - l^2}}$$

$$\frac{d^2A}{dl^2} = \frac{\sqrt{4a^2 - l^2}(-4l) - (4a^2 - 2l^2)\frac{(-2l)}{2\sqrt{4a^2 - l^2}}}{4a^2 - l^2}$$
$$= \frac{(4a^2 - l^2)(-4l) + l(4a^2 - l^2)}{(4a^2 - l^2)^{\frac{3}{2}}}$$
$$= \frac{-12a^2l + 2l^3}{(4a^2 - l^2)^{\frac{3}{2}}}$$
$$= \frac{-2l(6a^2 - l^2)}{(4a^2 - l^2)^{\frac{3}{2}}}$$

Now,

$$\frac{dA}{dl} = 0$$

Hence,

$$\Rightarrow \frac{4a^2 - 2l^2}{\sqrt{4a^2 - l^2}} = 0$$
$$\Rightarrow 4a^2 = 2l^2$$
$$\Rightarrow l = \sqrt{2}a$$

Thus,

$$
\begin{aligned}
b &= \sqrt{4a^2 - 2a^2} \\
&= \sqrt{2a^2} \\
&= \sqrt{2}a
\end{aligned}
$$

When, $l = \sqrt{2}a$
Then,

$$
\begin{aligned}
\frac{d^2A}{dl^2} &= \frac{-2\left(\sqrt{2}a\right)\left(6a^2 - 2a^2\right)}{2\sqrt{2}a^3} \\
&= \frac{-8\sqrt{2}a^3}{2\sqrt{2}a^3} \\
&= -4 < 0
\end{aligned}
$$

By the second derivative test, when $l = \sqrt{2}a$, then the area of the rectangle is the maximum.

Since, $l = b = \sqrt{2}a$ the rectangle is a square.

Hence, it has been proved that of all the rectangles inscribed in the given fixed circle, the square has the maximum area.

Question 20:

Show that the right circular cylinder of given surface and maximum volume is such that is heights is equal to the diameter of the base.

Solution:

Let r and h be the radius and height of the cylinder respectively.

Then, the surface area (S) of the cylinder is given by,

$$S = 2\pi r^2 + 2\pi rh$$

Therefore,

$$
\begin{aligned}
h &= \frac{S - 2\pi r^2}{2\pi r} \\
&= \frac{S}{2\pi}\left(\frac{1}{r}\right) - r
\end{aligned}
$$

Let V be the volume of the cylinder.

Then,

$$V = \pi r^2 h$$
$$= \pi r^2 \left[\frac{S}{2\pi} \left(\frac{1}{r} \right) - r \right]$$
$$= \frac{Sr}{2} - \pi r^3$$
$$\frac{dV}{dr} = \frac{S}{2} - 3\pi r^2$$
$$\frac{d^2V}{dr^2} = -6\pi r$$

Now,

$$\frac{dV}{dr} = 0$$
$$\Rightarrow \frac{S}{2} - 3\pi r^2 = 0$$
$$\Rightarrow \frac{S}{2} = 3\pi r^2$$
$$\Rightarrow r^2 = \frac{S}{6\pi}$$

When, $r^2 = \frac{S}{6\pi}$
Then,

$$\frac{d^2V}{dr^2} = -6\pi \left(\sqrt{\frac{S}{6\pi}} \right) < 0$$

By second derivative test, the volume is the maximum when $r^2 = \frac{S}{6\pi}$.

Now, when $r^2 = \frac{S}{6\pi}$,
Then,

$$h = \frac{6\pi r^2}{2\pi} \left(\frac{1}{r} \right) - r$$
$$= 3r - r$$
$$= 2r$$

Hence, the volume is the maximum when the height is twice the radius i.e., when the height is equal to the diameter.

Question 21:

Of all the closed cylindrical cans (right circular), of a given volume of 100 cubic centimeters, find the dimensions of the can which has the minimum surface area?

Solution:

Let r and h be the radius and height of the cylinder respectively.

Then, volume (V) of the cylinder is given by,

$$V = \pi r^2 h = 100$$
$$\Rightarrow h = \frac{100}{\pi r^2}$$

Surface area (S) is given by:

$$S = 2\pi r^2 + 2\pi rh$$
$$= 2\pi r^2 + \frac{200}{r}$$

Hence,

$$\frac{dS}{dr} = 4\pi r - \frac{200}{r^2}$$
$$\frac{d^2S}{dr^2} = 4\pi + \frac{400}{r^3}$$

Now,

$$\frac{dS}{dr} = 0$$
$$\Rightarrow 4\pi r - \frac{200}{r^2} = 0$$
$$\Rightarrow 4\pi r = \frac{200}{r^2}$$
$$\Rightarrow r^3 = \frac{200}{4\pi} = \frac{50}{\pi}$$
$$\Rightarrow r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$$

When, $r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$

Then,

$$\frac{d^2S}{dr^2} > 0$$

By second derivative test, the surface area is the minimum when the radius of the cylinder is $\left(\frac{50}{\pi}\right)^{\frac{1}{3}} cm$.

When, $r=\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$

Then,

$$h=\frac{100}{\pi\left(\frac{50}{\pi}\right)^{\frac{2}{3}}}$$

$$=\frac{2\times 50}{(50)^{\frac{2}{3}}(\pi)^{1-\frac{2}{3}}}$$

$$=2\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$$

Hence, the required dimensions of the can which has the minimum surface area is given by radius $\left(\frac{50}{\pi}\right)^{\frac{1}{3}} cm$ and height $2\left(\frac{50}{\pi}\right)^{\frac{1}{3}} cm$.

Question 22:

A wire of length $28m$ is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that the combined area of the square and the circle is minimum?

Solution:

Let a piece of length l be cut from the given wire to make a square.

Then, the other piece of wire to be made into a circle is of length $(28-l)m$.

Now, side of square is $\frac{l}{4}$

Let r be the radius of the circle.

Then,

$$2\pi r=28-l$$

$$\Rightarrow r=\frac{l}{2\pi}(28-l)$$

The combined areas of the square and the circle, (A) is given by,

$$A = (side\ of\ the\ square)^2 + \pi r^2$$
$$= \frac{l^2}{16} + \pi\left[\frac{1}{2\pi}(28-l)\right]^2$$
$$= \frac{l^2}{16} + \frac{1}{4\pi}(28-l)^2$$

Hence,

$$\frac{dA}{dl} = \frac{2l}{16} + \frac{2}{4\pi}(28-l)(-1)$$
$$= \frac{1}{8} - \frac{1}{2\pi}(28-l)$$
$$\frac{d^2A}{dl^2} = \frac{1}{8} + \frac{1}{2\pi} > 0$$

Now,

$$\frac{d^2A}{dl^2} = 0$$
$$\Rightarrow \frac{l}{8} - \frac{l}{2\pi}(28-l) = 0$$
$$\Rightarrow \frac{\pi l - 4(28-l)}{8\pi} = 0$$
$$\Rightarrow (\pi+4)l - 112 = 0$$
$$\Rightarrow l = \frac{112}{\pi+4}$$

When, $l = \frac{112}{\pi+4}$
Then,

$$\frac{d^2A}{dl^2} > 0$$

By second derivative test, the area (A) is the minimum when $l = \frac{112}{\pi+4}m$.

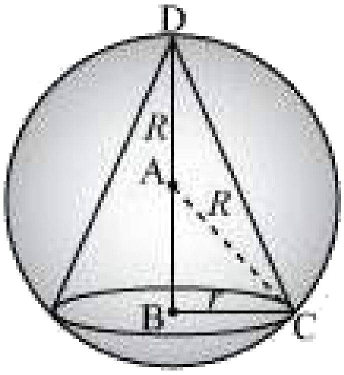
Hence, the combined area is the minimum when the length of the wire in making the square is $\frac{112}{\pi+4}m$ while the length of the wire in making the circle is $\left(28 - \frac{112}{\pi+4}\right) = \frac{28\pi}{\pi+4}m$.

Question 23:

Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.

Solution:

Let r and h be the radius and height of the cone respectively inscribed in a sphere of radius R.



Let V be the volume of the cone.

Then, $V = \frac{1}{3}\pi r^2 h$

Height of the cone is given by,

$$h = R + AB$$
$$= R + \sqrt{R^2 - r^2} \qquad [ABC \text{ is a right triangle}]$$

Hence,

$$V = \frac{1}{3}\pi r^2 \left(R + \sqrt{R^2 - r^2}\right)$$
$$= \frac{1}{3}\pi r^2 R + \frac{1}{3}\pi r^2 \sqrt{R^2 - r^2}$$

Therefore,

$$\frac{dV}{dr} = \frac{2}{3}\pi rR + \frac{2}{3}\pi r\sqrt{R^2 - r^2} - \frac{1}{3}\pi r^2 . \frac{(-2r)}{\sqrt{R^2 - r^2}}$$

$$= \frac{2}{3}\pi rR + \frac{2}{3}\pi r\sqrt{R^2 - r^2} - \frac{1}{3}\pi . \frac{r^3}{\sqrt{R^2 - r^2}}$$

$$= \frac{2}{3}\pi rR + \frac{2\pi r\left(R^2 - r^2\right) - \pi r^3}{3\sqrt{R^2 - r^2}}$$

$$= \frac{2}{3}\pi rR + \frac{2\pi rR^2 - 3\pi r^3}{3\sqrt{R^2 - r^2}}$$

$$\frac{d^2V}{dr^2} = \frac{2}{3}\pi R + \frac{3\sqrt{R^2 - r^2}\left(2\pi R^2 - 9\pi r^2\right) - \left(2\pi rR^2 - 3\pi r^3\right) . \frac{(-2r)}{6\sqrt{R^2 - r^2}}}{9\left(R^2 - r^2\right)}$$

$$= \frac{2}{3}\pi R + \frac{9\left(R^2 - r^2\right)\left(2\pi R^2 - 9\pi r^2\right) + 2\pi r^2 R^2 + 3\pi r^4}{27\left(R^2 - r^2\right)^{\frac{3}{2}}}$$

Now,

$$\frac{dV}{dr} = 0$$

$$\Rightarrow \frac{2}{3}\pi rR + \frac{2\pi rR^2 - 3\pi r^3}{3\sqrt{R^2 - r^2}} = 0$$

$$\Rightarrow \frac{2}{3}\pi rR = \frac{3\pi r^3 - 2\pi rR^2}{3\sqrt{R^2 - r^2}}$$

$$\Rightarrow 2R = \frac{3r^2 - 2R^2}{\sqrt{R^2 - r^2}}$$

$$\Rightarrow 2R\sqrt{R^2 - r^2} = 3r^2 - 2R^2$$

$$\Rightarrow 4R^2\left(R^2 - r^2\right) = \left(3r^2 - 2R^2\right)^2$$

$$\Rightarrow 4R^4 - 4R^2r^2 = 9r^4 + 4R^4 - 12r^2R^2$$

$$\Rightarrow 9r^4 = 8R^2r^2$$

$$\Rightarrow r^2 = \frac{8}{9}R^2$$

When, $r^2 = \frac{8}{9}R^2$

Then, $\frac{dV}{dr^2} < 0$

By second derivative test, the volume of the cone is the maximum when $r^2 = \frac{8}{9}R^2$.

When, $r^2 = \frac{8}{9}R^2$

Then,

$$
\begin{aligned}
h &= R + \sqrt{R^2 - \frac{8}{9}R^2} \\
&= R + \sqrt{\frac{1}{9}R^2} \\
&= R + \frac{R}{3} \\
&= \frac{4}{3}R
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
V &= \frac{1}{3}\pi\left(\frac{8}{9}R^2\right)\left(\frac{4}{3}R\right) \\
&= \frac{8}{27}\left(\frac{4}{3}\pi R^3\right) \\
&= \frac{8}{27}\times(\text{Volume of sphere})
\end{aligned}
$$

Hence, the volume of the largest cone that can be inscribed in the sphere is $\frac{8}{27}$ the volume of the sphere.

Question 24:

Show that the right circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ time the radius of the base.

Solution:

Let r and h be the radius and height of the cone, respectively.

Then, the volume (V) of the cone is given by,

$$
\begin{aligned}
V &= \frac{1}{3}\pi r^2 h \\
\Rightarrow h &= \frac{3V}{\pi r^2}
\end{aligned}
$$

The surface area (S) of the cone is given by,

$S = \pi r l$, where l is the slant height

Hence,

$$
\begin{aligned}
S &= \pi r\sqrt{r^2+h^2} \\
&= \pi r\sqrt{r^2+\frac{9V^2}{\pi^2 r^4}} \\
&= \frac{\pi r\sqrt{\pi^2 r^6+9V^2}}{\pi r^2} \\
&= \frac{1}{r}\sqrt{\pi^2 r^6+9V^2}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{dS}{dr} &= \frac{r.\dfrac{6\pi^2 r^5}{2\sqrt{\pi^2 r^6+9V^2}}-\sqrt{\pi^2 r^6+9V^2}}{r^2} \\
&= \frac{3\pi^2 r^6-\pi^2 r^6-9V^2}{r^2\sqrt{\pi^2 r^6+9V^2}} \\
&= \frac{2\pi^2 r^6-9V^2}{r^2\sqrt{\pi^2 r^6+9V^2}}
\end{aligned}
$$

Now,

$$
\begin{aligned}
&\frac{dS}{dr}=0 \\
&\Rightarrow \frac{2\pi^2 r^6-9V^2}{r^2\sqrt{\pi^2 r^6+9V^2}}=0 \\
&\Rightarrow 2\pi^2 r^6 = 9V^2 \\
&\Rightarrow r^6 = \frac{9V^2}{2\pi^2}
\end{aligned}
$$

Thus, it can be easily verified that when $r^6=\frac{9V^2}{2\pi^2}$, $\Rightarrow \frac{d^2S}{dr^2}>0$

By second derivative test, the surface area of the cone is the least when $r^6=\frac{9V^2}{2\pi^2}$.

When, $r^6=\frac{9V^2}{2\pi^2}$

Then,

$$
\begin{aligned}
h &= \frac{3V}{\pi r^2} \\
&= \frac{3V}{\pi r^2}\left(\frac{2\pi^2 r^6}{9}2\right)^{\frac{1}{2}} \\
&= \frac{3}{\pi r^2}.\frac{\sqrt{2}\pi r^3}{3} \\
&= \sqrt{2}r
\end{aligned}
$$

Hence, for a given volume, the right circular cone of the least curved surface has an altitude equal to $\sqrt{2}$ times the radius of the base.

Question 25:

Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$.

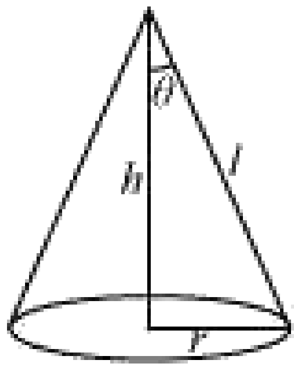
Solution:

Let θ be the semi-vertical angle of the cone.

It is clear that $\theta \in \left[0, \frac{\pi}{2}\right]$.

Let r, h and l be the radius, height, and the slant height of the cone respectively.

The slant height of the cone is given as constant.



Now, $r = l\sin\theta$ and $h = l\cos\theta$

The volume (V) of the cone is given by,

$$V = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi\left(l^2 \sin^2\theta\right)\left(l \cos\theta\right)$$
$$= \frac{1}{3}\pi l^3 \sin^2\theta\cos\theta$$

Therefore,

$$
\begin{aligned}
\frac{dV}{d\theta} &= \frac{\pi l^3}{3}\left[\sin^2\theta\left(-\sin\theta\right)+\cos\theta\left(2\sin\theta\cos\theta\right)\right] \\
&= \frac{\pi l^3}{3}\left[-\sin^3\theta+2\sin\theta\cos^3\theta\right] \\
\frac{d^2V}{d\theta^2} &= \frac{\pi l^3}{3}\left[-3\sin^2\theta\cos\theta+2\cos^3\theta-4\sin^2\theta\cos\theta\right] \\
&= \frac{\pi l^3}{3}\left[2\cos^3\theta-7\sin^2\theta\cos\theta\right]
\end{aligned}
$$

Now,

$$\frac{dV}{d\theta}=0$$
$$\Rightarrow \frac{\pi l^3}{3}\left[-\sin^3\theta+2\sin\theta\cos^3\theta\right]=0$$
$$\Rightarrow \sin^3\theta=2\sin\theta\cos^3\theta$$
$$\Rightarrow \tan^2\theta=2$$
$$\Rightarrow \tan\theta=\sqrt{2}$$
$$\Rightarrow \theta=\tan^{-1}\sqrt{2}$$

When, $\theta=\tan^{-1}\sqrt{2}$
Then, $\tan^2\theta=2$ or $\sin^2\theta=2\cos^2\theta$.

Hence, we have:

$$\frac{d^2V}{d\theta^2}=\frac{\pi l^3}{3}\left[2\cos^3\theta-14\cos^3\theta\right]$$
$$=\frac{\pi l^3}{3}\left[-12\cos^3\theta\right]$$
$$=-4\pi l^3\cos^3\theta<0 \quad \forall \quad \theta\in\left[0,\frac{\pi}{2}\right]$$

By second derivative test, the volume (V) is the maximum when $\theta=\tan^{-1}\sqrt{2}$.

Hence, for a given slant height, the semi-vertical angle of the cone of the maximum volume is $\tan^{-1}\sqrt{2}$.

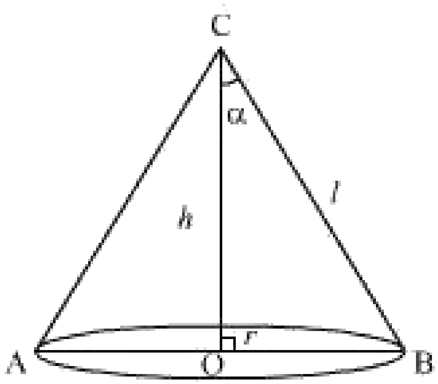
Question 26:

Show that semi-vertical angle of right circular cone of given surface area and maximum volume is $\sin^{-1}\left(\frac{1}{3}\right)$.

Solution:

Let r be the radius, l be the slant height and h be the height of the cone of given surface area S.

Also, let α be the semi-vertical angle of the cone.



Then,

$$S=\pi rl+\pi r^2$$
$$\Rightarrow l=\frac{S-\pi r^2}{\pi r} \qquad \ldots(1)$$

Let V be the volume of the cone.

Then

$$V=\frac{1}{3}\pi r^2 h$$
$$V^2=\frac{1}{9}\pi^2 r^4 h^2$$
$$=\frac{1}{9}\pi^2 r^4\left(l^2-r^2\right) \qquad \left[\because l^2=r^2+h^2\right]$$
$$=\frac{1}{9}\pi^2 r^4\left[\left(\frac{S-\pi r^2}{\pi r}\right)^2-r^2\right]$$
$$=\frac{1}{9}\pi^2 r^4\left[\frac{\left(S-\pi r^2\right)^2-\pi^2 r^4}{\pi^2 r^2}\right]$$
$$=\frac{1}{9}r^2\left(S^2-2S\pi r^2\right)$$
$$V^2=\frac{1}{9}Sr^2\left(S^2-2\pi r^2\right) \qquad \ldots(2)$$

Differentiating (2) with respect to r, we get

$$2V\frac{dV}{dr}=\frac{1}{9}S\left(2Sr-8\pi r^3\right)$$

For maximum or minimum, put $\frac{dV}{dr}=0$

$$\Rightarrow \frac{1}{9}S\left(2Sr-8\pi r^3\right)=0$$

$$\Rightarrow 2Sr-8\pi r^3=0 \qquad [\because S\neq 0]$$

$$\Rightarrow S=4\pi r^2 \qquad [\because r\neq 0]$$

$$\Rightarrow r^2=\frac{S}{4\pi}$$

Differentiating again with respect to r, we get

$$2V\frac{d^2V}{dr^2}+2\left(\frac{dV}{dr}\right)^2=\frac{1}{9}S\left(2S-24\pi r^2\right)$$

$$\Rightarrow 2V\frac{d^2V}{dr^2}=\frac{1}{9}S\left(2S-24\pi\times\frac{S}{4\pi}\right) \qquad \left[\because \frac{dV}{dr}=0 \text{ and } r^2=\frac{S}{4\pi}\right]$$

$$\Rightarrow 2V\frac{d^2V}{dr^2}=\frac{1}{9}S\left(2S-6S\right)$$

$$\Rightarrow 2V\frac{d^2V}{dr^2}=-\frac{4}{9}S^2<0$$

Thus, V is maximum when $S=4\pi r^2$.

Therefore,

$$S=\pi rl+\pi r^2$$

$$\Rightarrow 4\pi r^2=\pi rl+\pi r^2$$

$$\Rightarrow 3\pi r^2=\pi rl$$

$$\Rightarrow l=3r$$

Now, in ΔCOB,

$$\sin\alpha=\frac{OB}{BC}$$

$$=\frac{r}{l}$$

$$=\frac{r}{3r}=\frac{1}{3}$$

$$\alpha=\sin^{-1}\left(\frac{1}{3}\right)$$

Question 27:

The point on the curve $x^2=2y$ which is nearest to the point $(0,5)$ is

(A) $\left(2\sqrt{2},4\right)$ **(B)** $\left(2\sqrt{2},0\right)$ **(C)** $(0,0)$ **(D)** $(2,2)$

Solution:

The given curve is $x^2 = 2y$.

For each value of x, the position of the point will be $\left(x, \frac{x^2}{2}\right)$.

Let $P\left(x, \frac{x^2}{2}\right)$ and $A(0,5)$ are the given points.

Now distance between the points P and A is given by,

$$\Rightarrow PA = \sqrt{(x-0)^2 + \left(\frac{x^2}{2} - 5\right)^2}$$

$$\Rightarrow PA^2 = (x-0)^2 + \left(\frac{x^2}{2} - 5\right)^2$$

$$\Rightarrow PA^2 = x^2 + \frac{x^4}{4} + 25 - 5x^2$$

$$\Rightarrow PA^2 = \frac{x^4}{4} - 4x^2 + 25$$

$$\Rightarrow PA^2 = y^2 - 8y + 25 \qquad (\because x^2 = 2y)$$

Let us denote PA^2 by Z

Then, $Z = y^2 - 8y + 25$

Differentiating both sides with respect to y, we get

$$\frac{dZ}{dy} = 2y - 8$$

For maxima or minima, we have

$$\frac{dZ}{dy} = 0$$
$$\Rightarrow 2y - 8 = 0$$
$$\Rightarrow 2y = 8$$
$$\Rightarrow y = 4$$

Also,

$$\frac{d^2Z}{dy^2} = 2$$

Now,

$$\left[\frac{d^2Z}{dy^2}\right]_{y=4} = 2 > 0$$

$$\Rightarrow x^2 = 2y$$

$$\Rightarrow x^2 = 2\times 4$$

$$\Rightarrow x^2 = 8$$

$$\Rightarrow x = \pm 2\sqrt{2}$$

So, Z is minimum at $\left(2\sqrt{2},4\right)$ or $\left(-2\sqrt{2},4\right)$.

Or, PA^2 is minimum at $\left(2\sqrt{2},4\right)$ or $\left(-2\sqrt{2},4\right)$.

Or, PA is minimum at $\left(2\sqrt{2},4\right)$ or $\left(-2\sqrt{2},4\right)$.

So, distance between the points $P\left(x,\frac{x^2}{2}\right)$ and $A(0,5)$ is minimum at $\left(2\sqrt{2},4\right)$ or $\left(-2\sqrt{2},4\right)$.

Thus, the correct option is **A**.

Question 28:

For all real values of x, the minimum value of $\frac{1-x+x^2}{1+x+x^2}$ is

(A) 0 **(B)** 1 **(C)** 3 **(D)** $\frac{1}{3}$

Solution:

Let $f(x) = \frac{1-x+x^2}{1+x+x^2}$

Therefore,

$$f'(x)=\frac{\left(1+x+x^2\right)\left(-1+2x\right)-\left(1-x+x^2\right)\left(1+2x\right)}{\left(1+x+x^2\right)^2}$$
$$=\frac{-1+2x-x+2x^2-x^2+2x^3-1-2x+x+2x^2-x^2-2x^3}{\left(1+x+x^2\right)^2}$$
$$=\frac{2x^2-2}{\left(1+x+x^2\right)^2}$$
$$=\frac{2\left(x^2-1\right)}{\left(1+x+x^2\right)^2}$$

Now,

$$f'(x)=0$$
$$\Rightarrow x^2=1$$
$$\Rightarrow x=\pm 1$$

Also,

$$f''(x)=\frac{2\left[\left(1+x+x^2\right)^2(2x)-\left(x^2-1\right)(2)\left(1+x+x^2\right)\left(1+2x\right)\right]}{\left(1+x+x^2\right)^4}$$
$$=\frac{4\left(1+x+x^2\right)\left[\left(1+x+x^2\right)x-\left(x^2-1\right)\left(1+2x\right)\right]}{\left(1+x+x^2\right)^4}$$
$$=\frac{4\left[x+x^2+x^3-x^2-2x^3+1+2x\right]}{\left(1+x+x^2\right)^3}$$
$$=\frac{4\left(1+3x-x^3\right)}{\left(1+x+x^2\right)^3}$$

Hence,

$$f''(1)=\frac{4\left(1+3-1\right)}{\left(1+1+1\right)^3}$$
$$=\frac{4(3)}{(3)^3}$$
$$=\frac{12}{27}$$
$$=\frac{4}{9}>0$$

Also,

$$f''(-1) = \frac{4(1-3-1)}{(1-1+1)^3}$$
$$= 4(-1)$$
$$= -4 < 0$$

By second derivative test, f is the minimum at $x = 1$ and the minimum value is given by,

$$f(1) = \frac{1-1+1}{1+1+1}$$
$$= \frac{1}{3}$$

Thus, the correct option is **D**.

Question 29:

The maximum value of $\left[x(x-1)+1\right]^{\frac{1}{3}}, 0 \le x \le 1$ is

(A) $\left(\frac{1}{3}\right)^{\frac{1}{3}}$ **(B)** $\frac{1}{2}$ **(C)** 1 **(D)** 0

Solution:

Let $f(x) = \left[x(x-1)+1\right]^{\frac{1}{3}}$

Therefore,

$$f'(x) = \frac{2x-1}{3\left[x(x-1)+1\right]^{\frac{2}{3}}}$$

Now,

$$f'(x) = 0$$
$$\Rightarrow x = \frac{1}{2}$$

Then, we evaluate the value of f at critical point $x = \frac{1}{2}$ and at the end points of the interval $[0,1]$ i.e., at $x = 0$ and $x = 1$.

$$f(0)=[0(0-1)+1]^{\frac{1}{3}}$$
$$=1$$
$$f(1)=[1(1-1)+1]^{\frac{1}{3}}$$
$$=1$$
$$f\left(\frac{1}{2}\right)=\left[\frac{1}{2}\left(-\frac{1}{2}\right)+1\right]^{\frac{1}{3}}$$
$$=\left(\frac{3}{4}\right)^{\frac{1}{3}}$$

Hence, we can conclude that the maximum value of f in the interval $[0,1]$ is 1.

Thus, the correct option is **C**.

MISCELLANEOUS EXERCISE

Question 1:

Using differentials, find the approximate value of each of the following.

(i) $\left(\frac{17}{81}\right)^{\frac{1}{4}}$

(ii) $(33)^{-\frac{1}{5}}$

Solution:

(i) $\left(\frac{17}{81}\right)^{\frac{1}{4}}$

Consider $y=(x)^{\frac{1}{4}}$

Let $x=\frac{16}{81}$ and $\Delta x=\frac{1}{81}$

Then,

$$\Delta y=(x+\Delta x)^{\frac{1}{4}}-(x)^{\frac{1}{4}}$$
$$=\left(\frac{17}{81}\right)^{\frac{1}{4}}-\left(\frac{16}{81}\right)^{\frac{1}{4}}$$
$$=\left(\frac{17}{81}\right)^{\frac{1}{4}}-\frac{2}{3}$$
$$\frac{2}{3}+\Delta y=\left(\frac{17}{81}\right)^{\frac{1}{4}}$$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x$$

$$= \frac{1}{4(x)^{\frac{3}{4}}}(\Delta x) \qquad \left[\because y = (x)^{\frac{1}{4}}\right]$$

$$= \frac{1}{4\left(\frac{16}{81}\right)^{\frac{3}{4}}}\left(\frac{1}{81}\right)$$

$$= \frac{27}{4\times 8}\times\frac{1}{81}$$

$$= \frac{1}{32\times 3}$$

$$= \frac{1}{96}$$

$$= 0.010$$

Hence,

$$\left(\frac{17}{81}\right)^{\frac{1}{4}} = \frac{2}{3} + 0.010$$

$$= 0.667 + 0.010$$

$$= 0.677$$

Thus, the approximate value of $\left(\frac{17}{81}\right)^{\frac{1}{4}} = 0.677$.

(ii) $(33)^{-\frac{1}{5}}$

Consider $y = (x)^{-\frac{1}{5}}$

Let $x = 32$ and $\Delta x = 1$

Then,

$$\Delta y = (x + \Delta x)^{-\frac{1}{5}} - (x)^{-\frac{1}{5}}$$

$$= (33)^{-\frac{1}{5}} - (32)^{-\frac{1}{5}}$$

$$= (33)^{-\frac{1}{5}} - \frac{1}{2}$$

$$\frac{1}{2} + \Delta y = (33)^{-\frac{1}{5}}$$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x$$

$$= \frac{-1}{5(x)^{\frac{6}{5}}}(\Delta x) \qquad \left[\because y = (x)^{-\frac{1}{5}}\right]$$

$$= -\frac{1}{5(2)^{6}}(1)$$

$$= -\frac{1}{320}$$

$$= -0.003$$

Hence,

$$(33)^{-\frac{1}{5}} = \frac{1}{2} + (-0.003)$$

$$= 0.5 - 0.003$$

$$= 0.497$$

Thus, the approximate value of $(33)^{-\frac{1}{5}} = 0.497$.

Question 2:

Show that the function given by $f(x) = \frac{\log x}{x}$ has maximum at $x = e$.

Solution:

The given function is $f(x) = \frac{\log x}{x}$
Therefore,

$$f'(x) = \frac{x\left(\frac{1}{x}\right) - \log x}{x^2}$$

$$= \frac{1 - \log x}{x^2}$$

Now,

$$f'(x) = 0$$

$$\Rightarrow 1 - \log x = 0$$

$$\Rightarrow \log x = 1$$

$$\Rightarrow \log x = \log e$$

$$\Rightarrow x = e$$

Also,

$$f''(x) = \frac{x^2\left(-\frac{1}{x}\right) - (1-\log x)(2x)}{x^4}$$
$$= \frac{-x - 2x(1-\log x)}{x^4}$$
$$= \frac{-3 + 2\log x}{x^3}$$

Now,

$$f''(e) = \frac{-3 + 2\log e}{e^3}$$
$$= \frac{-3+2}{e^3}$$
$$= \frac{-1}{e^3} < 0$$

Therefore, by second derivative test, f is the maximum at $x = e$.

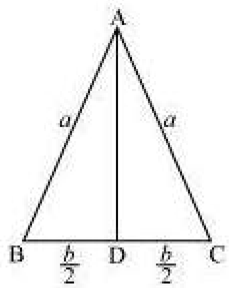
Question 3:

The two equal sides of an isosceles triangle with fixed base b are decreasing at the rate of 3 cm per second. How fast is the area decreasing when the two equal sides are equal to the base?

Solution:

Let ΔABC be isosceles where BC is the base of fixed length b.

Also, let the length of the two equal sides of ΔABC be a.
Draw $AD \perp BC$.



Now, in ΔADC, by applying the Pythagoras theorem, we have:

$$AD = \sqrt{a^2 - \frac{b^2}{4}}$$

Area of triangle,

$$A=\frac{1}{2}b\sqrt{a^2-\frac{b^2}{4}}$$

The rate of change of the area with respect to time (t) is given by,

$$\frac{dA}{dt}=\frac{1}{2}b.\frac{2a}{2\sqrt{a^2-\frac{b^2}{4}}}\frac{da}{dt}$$
$$=\frac{ab}{\sqrt{4a^2-b^2}}\frac{da}{dt}$$

It is given that the two equal sides of the triangle are decreasing at the rate of $3cm$ per second. Therefore,

$$\frac{da}{dt}=-3cm/s$$

Hence,

$$\Rightarrow\frac{dA}{dt}=\frac{-3ab}{\sqrt{4a^2-b^2}}$$

When, $a=b$ we have:

$$\frac{dA}{dt}=\frac{-3b^2}{\sqrt{4a^2-b^2}}$$
$$=\frac{-3b^2}{\sqrt{3b^2}}$$
$$=-\sqrt{3}b$$

Hence, if the two equal sides are equal to the base, then the area of the triangle is decreasing at the rate of $-\sqrt{3}bcm^2/s$.

Question 4:

Find the equation of the normal to curve $y^2=4x$ at the point $(1,2)$.

Solution:

The equation of the given curve is $y^2=4x$

Differentiating with respect to x, we have:

$$2y\frac{dy}{dx}=4$$
$$\Rightarrow \frac{dy}{dx}=\frac{4}{2y}$$
$$\Rightarrow \frac{dy}{dx}=\frac{2}{y}$$
$$\Rightarrow \left.\frac{dy}{dx}\right]_{(1,2)}=\frac{2}{2}=1$$

Now, the slope of the normal at point $(1,2)$ is

$$\frac{-1}{\left.\frac{dy}{dx}\right]_{(1,2)}}=\frac{-1}{1}=-1$$

Equation of the normal at $(1,2)$ is

$$\Rightarrow y-2=-x+1$$
$$\Rightarrow x+y-3=0$$

Question 5:

Show that the normal at any point θ to the curve $x=a\cos\theta+a\theta\sin\theta$, $y=a\sin\theta-a\theta\cos\theta$ is at a constant distance from the origin.

Solution:

We have $x=a\cos\theta+a\theta\sin\theta$
Therefore,

$$\frac{dx}{d\theta}=-a\sin\theta+a\sin\theta+a\theta\cos\theta$$
$$=a\theta\cos\theta$$

Also, $y=a\sin\theta-a\theta\cos\theta$
Hence,

$$\frac{dy}{d\theta}=a\cos\theta-a\cos\theta+a\theta\cos\theta$$
$$=a\theta\cos\theta$$

Thus,

$$\frac{dy}{dx}=\frac{dy}{d\theta}.\frac{d\theta}{dx}$$
$$=\frac{a\theta\sin\theta}{a\theta\cos\theta}$$
$$=\tan\theta$$

Slope of the normal at any point θ is $\frac{-1}{\tan\theta}$.

The equation of the normal at a given point (x, y) is given by,

$$y - a\sin\theta + a\theta\cos\theta = \frac{-1}{\tan\theta}(x - a\cos\theta - a\theta\sin\theta)$$

$$\Rightarrow y\sin\theta - a\sin^2\theta + a\theta\sin\theta\cos\theta = -x\cos\theta + a\cos^2\theta + a\theta\sin\theta\cos\theta$$

$$\Rightarrow x\cos\theta + y\sin\theta - a(\sin^2\theta + \cos^2\theta) = 0$$

$$\Rightarrow x\cos\theta + y\sin\theta - a = 0$$

Now, the perpendicular distance of the normal from the origin is

$$\frac{|-a|}{\sqrt{\cos^2\theta + \sin^2\theta}} = \frac{|-a|}{\sqrt{1}} = |-a|$$, which is independent of θ

Hence, the perpendicular distance of the normal from the origin is constant.

Question 6:

Find the intervals in which the function f given by $f(x) = \frac{4\sin x - 2x - x\cos x}{2 + \cos x}$ is

(i) Increasing

(ii) Decreasing

Solution:

We have $f(x) = \frac{4\sin x - 2x - x\cos x}{2 + \cos x}$

Hence,

$$f'(x) = \frac{(2+\cos x)(4\cos x - 2 - \cos x + x\sin x) - (4\sin x - 2x - x\cos x)(-\sin x)}{(2+\cos x)^2}$$

$$= \frac{6\cos x - 4 + 2x\sin x + 3\cos^2 x - 2\cos x + x\sin x\cos x + 4\sin^2 x - 2x\sin x - x\sin x\cos x}{(2+\cos x)^2}$$

$$= \frac{4\cos x - 4 + 3\cos^2 x + 4\sin^2 x}{(2+\cos x)^2}$$

$$= \frac{4\cos x - 4 + 3\cos^2 x + 4 - 4\cos^2 x}{(2+\cos x)^2}$$

$$= \frac{4\cos x - \cos^2 x}{(2+\cos x)^2}$$

$$f'(x)=\frac{\cos x(4-\cos x)}{(2+\cos x)^2}$$

Now,

$$f'(x)=0$$
$$\Rightarrow \cos x=0 \text{ or } \cos x=4$$

But $\cos x \neq 4$

Hence,

$$\cos x=0$$
$$\Rightarrow x=\frac{\pi}{2},\frac{3\pi}{2}$$

Now, $x=\frac{\pi}{2}$ and $x=\frac{3\pi}{2}$ divide $(0,2\pi)$ into three disjoint intervals i.e., $\left(0,\frac{\pi}{2}\right)$, $\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$ and $\left(\frac{3\pi}{2},2\pi\right)$.

In intervals, $\left(0,\frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2},2\pi\right)$, $f'(x)>0$

Thus, $f(x)$ is increasing for $0<x<\frac{x}{2}$ and $\frac{3x}{2}<x<2\pi$.

In the interval $\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$, $f'(x)<0$

Thus, $f(x)$ is decreasing for $\frac{\pi}{2}<x<\frac{3\pi}{2}$.

Question 7:

Find the intervals in which the function f given by $f(x)=x^3+\frac{1}{x^3}, x\neq 0$ is

(i) Increasing

(ii) Decreasing

Solution:

We have $f(x)=x^3+\frac{1}{x^3}$

Therefore,

$$f'(x) = 3x^2 - \frac{3}{x^4}$$
$$= \frac{3x^6 - 3}{x^4}$$

Now,

$$f'(x) = 0$$
$$\Rightarrow 3x^6 - 3 = 0$$
$$\Rightarrow x^6 = 1$$
$$\Rightarrow x = \pm 1$$

Now, the points $x = 1$ and $x = -1$ divide the real line into three disjoint intervals i.e., $(-\infty, -1)$, $(-1, 1)$ and $(1, \infty)$.

In intervals $(-\infty, -1)$ and $(1, \infty)$ i.e., when $x < -1$ and $x > 1$, $f'(x) > 0$.

Thus, when $x < -1$ and $x > 1$, f is increasing.

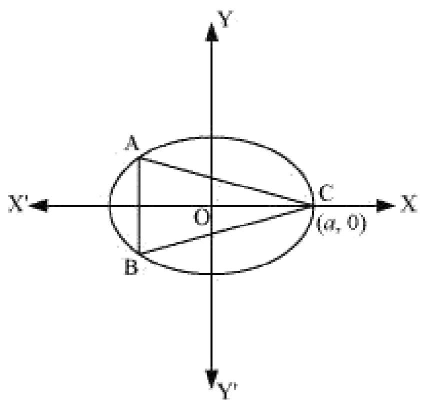
In interval $(-1, 1)$ i.e., when $-1 < x < 1$, $f'(x) < 0$.

Thus, when $-1 < x < 1$, f is decreasing.

Question 8:

Find the maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with its vertex at one end of the major axis.

Solution:



The given ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$
Let the major axis be along the x-axis.

Let ABC be the triangle inscribed in the ellipse where vertex C is at $(a,0)$.

Since the ellipse is symmetrical with respect to the x-axis and y-axis, we can assume the coordinates of A to be $(-x_1, y_1)$ and the coordinates of B to be $(-x_1,-y_1)$

Now, we have $y_1 = \pm\frac{b}{a}\sqrt{a^2-x_1^2}$
Coordinates of A are $\left(-x_1, \frac{b}{a}\sqrt{a^2-x_1^2}\right)$ and
the coordinates of B are $\left(x_1, -\frac{b}{a}\sqrt{a^2-x_1^2}\right)$

As the point (x_1, y_1) lies on the ellipse, the area of triangle ABC (A) is given by,

$$A=\frac{1}{2}\left|a\left(\frac{2b}{a}\sqrt{a^2-x_1^2}\right)+(-x_1)\left(-\frac{b}{a}\sqrt{a^2-x_1^2}\right)+(-x_1)\left(-\frac{b}{a}\sqrt{a^2-x_1^2}\right)\right|$$

$$=b\sqrt{a^2-x_1^2}+x_1\frac{b}{a}\sqrt{a^2-x_1^2} \qquad \ldots(1)$$

Therefore,

$$\frac{dA}{dx_1}=\frac{-2x_1b}{2\sqrt{a^2-x_1^2}}+\frac{b}{a}\sqrt{a^2-x_1^2}-\frac{-2bx_1^2}{2a\sqrt{a^2-x_1^2}}$$

$$=\frac{b}{a\sqrt{a^2-x_1^2}}\left[-x_1a+\left(a^2-x_1^2\right)-x_1^2\right]$$

$$=\frac{b\left(-2x_1^2-x_1a+a^2\right)}{a\sqrt{a^2-x_1^2}}$$

Now, $\frac{dA}{dx_1}=0$

Hence,

$$\Rightarrow -2x_1^{\ 2} - x_1 a + a^2 = 0$$

$$\Rightarrow x_1 = \frac{a \pm \sqrt{a^2 - 4(-2)(a^2)}}{2(-2)}$$

$$\Rightarrow x_1 = \frac{a \pm \sqrt{9a^2}}{-4}$$

$$\Rightarrow x_1 = \frac{a \pm 3a}{-4}$$

$$\Rightarrow x_1 = -a, \frac{a}{2}$$

But $x_1 \neq -a$

Therefore,

$$x_1 = \frac{a}{2}$$

$$y_1 = \frac{b}{a}\sqrt{a^2 - \frac{a^2}{4}}$$

$$= \frac{ba}{2a}\sqrt{3}$$

$$= \frac{\sqrt{3}b}{2}$$

Now,

$$\frac{d^2A}{dx_1^{\ 2}} = \frac{b}{a}\left\{\frac{\sqrt{a^2 - x_1^{\ 2}}\,(-4x_1 - a) - \left(-2x_1^{\ 2} - x_1 a + a^2\right)\frac{(-2x_1)}{2\sqrt{a^2 - x_1^{\ 2}}}}{a^2 - x_1^{\ 2}}\right\}$$

$$= \frac{b}{a}\left\{\frac{\left(a^2 - x_1^{\ 2}\right)(-4x_1 - a) + x_1\left(-2x_1^{\ 2} - x_1 a + a^2\right)}{\left(a^2 - x_1^{\ 2}\right)^{\frac{3}{2}}}\right\}$$

$$= \frac{b}{a}\left\{\frac{2x^3 - 3a^2 x - a^3}{\left(a^2 - x_1^{\ 2}\right)^{\frac{3}{2}}}\right\}$$

Also, when, $x_1 = \frac{a}{2}$

Then,

$$\frac{d^2A}{dx_1^2}=\frac{b}{a}\left\{\frac{2\left(\frac{a^3}{8}\right)-3a^2\left(\frac{a}{2}\right)-a^3}{\left(a^2-\left(\frac{a}{2}\right)^2\right)^{\frac{3}{2}}}\right\}$$

$$=\frac{b}{a}\left\{\frac{\frac{a^3}{4}-\frac{3}{2}a^3-a^3}{\left(\frac{3a^2}{4}\right)^{\frac{3}{2}}}\right\}$$

$$=-\frac{b}{a}\left\{\frac{\frac{9}{4}a^3}{\left(\frac{3a^2}{4}\right)^{\frac{3}{2}}}\right\}<0$$

Thus, the area is the maximum when $x_1=\frac{a}{2}$.

Hence, Maximum area of the triangle is given by,

$$A=b\sqrt{a^2-\frac{a^2}{4}}+\left(\frac{a}{2}\right)\frac{b}{a}\sqrt{a^2-\frac{a^2}{4}}$$

$$=ab\frac{\sqrt{3}}{2}+\left(\frac{a}{2}\right)\frac{b}{a}\times\frac{a\sqrt{3}}{2}$$

$$=\frac{ab\sqrt{3}}{2}+\frac{ab\sqrt{3}}{4}$$

$$=\frac{3\sqrt{3}}{4}ab$$

Question 9:

A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is $2m$ and volume is $8m^3$. If building of tank costs ₹ 70 per sq. meters for the base and ₹ 45 per square metres for sides. What is the cost of least expensive tank?

Solution:

Let l, b and h represent the length, breadth, and height of the tank respectively.

Then, we have height, $h=2m$ and volume of the tank, $V=8m^3$

Volume of the tank

$$V = lbh$$
$$\Rightarrow 8 = l \times b \times 2$$
$$\Rightarrow lb = 4$$
$$\Rightarrow b = \frac{4}{l}$$

Now, area of the base, $lb = 4$

Area of the 4 walls,

$$A = 2h(l + b)$$
$$= 4\left(l + \frac{4}{l}\right)$$

Hence,

$$\frac{dA}{dl} = 4\left(l - \frac{4}{l^2}\right)$$

Now,

$$\frac{dA}{dl} = 0$$
$$\Rightarrow \left(l - \frac{4}{l^2}\right) = 0$$
$$\Rightarrow l^2 = 4$$
$$\Rightarrow l = \pm 2$$

However, the length cannot be negative.

Therefore, we have $l = 4$

Hence,

$$b = \frac{4}{l}$$
$$= \frac{4}{2}$$
$$= 2$$

Now,

$$\frac{d^2 A}{dl^2} = \frac{32}{l^3}$$

When, $l = 2$

Then,

$$\frac{d^2A}{dl^2} = \frac{32}{8}$$
$$= 4 > 0$$

Thus, by second derivative test, the area is the minimum when $l = 2$

We have $l = b = h = 2$

Therefore,

Cost of building the base in ₹ is

$$70 \times (lb) = 70(4)$$
$$= 280$$

Cost of building the walls in ₹ is

$$2h(l + b) \times 45 = 2 \times 2(2 + 2) \times 45$$
$$= 720$$

Required total cost is ₹ is

$$280 + 720 = 1000$$

Thus, the total cost of the tank will be ₹ 1000.

Question 10:

The sum of the perimeter of a circle and square is k , where k is some constant. Prove that the sum of their areas is least when the side of square is double the radius of the circle.

Solution:

Let r be the radius of the circle and a be the side of the square.

Then, we have:

$$2\pi r + 4a = k$$
$$\Rightarrow a = \frac{k - 2\pi r}{4}$$

The sum of the areas of the circle and the square (A) is given by,

$$A = \pi r^2 + a^2$$
$$= \pi r^2 + \frac{(k - 2\pi r)^2}{16}$$

Hence,

$$\frac{dA}{dr} = 2\pi r + \frac{2(k-2\pi r)(-2\pi)}{16}$$
$$= 2\pi r - \frac{\pi(k-2\pi r)}{4}$$

Now,

$$\frac{dA}{dr} = 0$$
$$\Rightarrow 2\pi r - \frac{\pi(k-2\pi r)}{4} = 0$$
$$\Rightarrow 2\pi r = \frac{\pi(k-2\pi r)}{4}$$
$$\Rightarrow 8r = k - 2\pi r$$
$$\Rightarrow (8+2\pi)r = k$$
$$\Rightarrow r = \frac{k}{(8+2\pi)}$$
$$\Rightarrow r = \frac{k}{2(4+\pi)}$$

When, $r = \frac{k}{2(4+\pi)}$, $\Rightarrow \frac{d^2A}{dr^2} > 0$

The sum of the areas is least when, $r = \frac{k}{2(4+\pi)}$

When, $r = \frac{k}{2(4+\pi)}$
Then,

$$a = \frac{k - 2\pi\left[\frac{k}{2(4+\pi)}\right]}{4}$$
$$= \frac{k(4+\pi) - \pi k}{4(4+\pi)}$$
$$= \frac{4k}{4(4+\pi)}$$
$$= \frac{k}{4+\pi}$$
$$= 2r$$

Hence, it has been proved that the sum of their areas is least when the side of the square is double the radius of the circle.

Question 11:

A window is in the form of rectangle surmounted by a semicircular opening. The total perimeter of the window is $10m$. Find the dimensions of the window to admit maximum light through the whole opening.

Solution:

Let x and y be the length and breadth of the rectangular window.

Radius of the semicircular opening be $\frac{x}{2}$

It is given that the perimeter of the window is $10m$.

Therefore,

$$x+2y+\frac{\pi x}{2}=10$$
$$\Rightarrow x\left(1+\frac{\pi}{2}\right)+2y=10$$
$$\Rightarrow 2y=10-x\left(1+\frac{\pi}{2}\right)$$
$$\Rightarrow y=5-x\left(\frac{1}{2}+\frac{\pi}{4}\right)$$

Area of the window (A) is given by,

$$A=xy+\frac{\pi}{2}\left(\frac{x}{2}\right)^2$$
$$=x\left[5-x\left(\frac{1}{2}+\frac{\pi}{4}\right)\right]+\frac{\pi}{8}x^2$$
$$=5x-x^2\left(\frac{1}{2}+\frac{\pi}{4}\right)+\frac{\pi}{8}x^2$$

Therefore,

$$\frac{dA}{dx} = 5 - 2x\left(\frac{1}{2} + \frac{\pi}{4}\right) + \frac{\pi}{4}x$$
$$= 5 - x\left(1 + \frac{\pi}{2}\right) + \frac{\pi}{4}x$$
$$\frac{d^2A}{dx^2} = -\left(1 + \frac{\pi}{2}\right) + \frac{\pi}{4}$$
$$= -1 - \frac{\pi}{4}$$

Now,

$$\frac{dA}{dx} = 0$$
$$\Rightarrow 5 - x\left(1 + \frac{\pi}{2}\right) + \frac{\pi}{4}x = 0$$
$$\Rightarrow 5 - x - \frac{\pi}{4}x = 0$$
$$\Rightarrow x\left(1 + \frac{\pi}{4}\right) = 5$$
$$\Rightarrow x = \frac{5}{\left(1 + \frac{\pi}{4}\right)}$$
$$\Rightarrow x = \frac{20}{\pi + 4}$$

When, $x = \frac{20}{\pi + 4}$

Then, $\frac{d^2A}{dx^2} < 0$

Therefore, by second derivative test, the area is the maximum when length is $\frac{20}{\pi + 4}m$

Now,

$$y = 5 - \frac{20}{\pi + 4}\left(\frac{2 + \pi}{4}\right)$$
$$= 5 - \frac{5(2 + \pi)}{\pi + 4}$$
$$= \frac{10}{\pi + 4}$$

Hence, the required dimensions of the window to admit maximum light is given by length $\frac{20}{\pi+4}m$ and breadth $\frac{10}{\pi+4}m$.

Question 12:

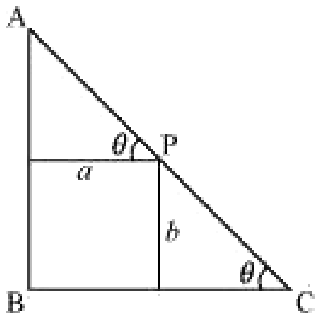
A point on the hypotenuse of a triangle is at distance *a* and *b* from the sides of the triangle.

Show that the minimum length of the hypotenuse is $\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^{\frac{3}{2}}$.

Solution:

Let ΔABC be right-angled at B, $AB=x$, $BC=y$ and $\angle C=\theta$.

Also, let P be a point on the hypotenuse of the triangle such that P is at a distance of a and b from the sides AB and BC respectively.



We have, $AC=\sqrt{x^2+y^2}$

Now,

$$PC = b\operatorname{cosec}\theta$$
$$AP = a\sec\theta$$

Hence,

$$AC = AP + PC$$
$$AC = b\operatorname{cosec}\theta + a\sec\theta \qquad \ldots(1)$$

Therefore,

$$\frac{d(AC)}{d\theta} = -b\operatorname{cosec}\theta\cot\theta + a\sec\theta\tan\theta$$

Now,

$$\frac{d(AC)}{d\theta}=0$$

$$\Rightarrow -b\cos ec\theta\cot\theta + a\sec\theta\tan\theta = 0$$

$$\Rightarrow a\sec\theta\tan\theta = b\cos ec\theta\cot\theta$$

$$\Rightarrow \frac{a}{\cos\theta}\cdot\frac{\sin\theta}{\cos\theta}=\frac{b}{\sin\theta}\cdot\frac{\cos\theta}{\sin\theta}$$

$$\Rightarrow a\sin^3\theta = b\cos^3\theta$$

$$\Rightarrow (a)^{\frac{1}{3}}\sin\theta = (b)^{\frac{1}{3}}\cos\theta$$

$$\Rightarrow \tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$$

Thus,

$$\sin\theta = \frac{(b)^{\frac{1}{3}}}{\sqrt{a^{\frac{2}{3}}+b^{\frac{2}{3}}}} \text{ and } \cos\theta = \frac{(a)^{\frac{1}{3}}}{\sqrt{a^{\frac{2}{3}}+b^{\frac{2}{3}}}} \qquad \ldots(2)$$

It can be clearly shown that $\frac{d^2(AC)}{d\theta^2}<0$, when $\tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$.

By second derivative test the length of the hypotenuse is the maximum when $\tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$

When, $\tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$, we have:

$$AC = \frac{b\sqrt{a^{\frac{2}{3}}+b^{\frac{2}{3}}}}{(b)^{\frac{1}{3}}}+\frac{a\sqrt{a^{\frac{2}{3}}+b^{\frac{2}{3}}}}{(a)^{\frac{1}{3}}} \qquad [\text{Using (1) and (2)}]$$

$$=\sqrt{a^{\frac{2}{3}}+b^{\frac{2}{3}}}\left(b^{\frac{2}{3}}+a^{\frac{2}{3}}\right)$$

$$=\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^{\frac{3}{2}}$$

Thus, the maximum length of the hypotenuses is $\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^{\frac{3}{2}}$ proved.

Question 13:

Find the points at which the function f given by $f(x)=(x-2)^4(x+1)^3$ has

(i) local maxima
(ii) local minima
(iii) point of inflexion

Solution:

The given function is $f(x)=(x-2)^4(x+1)^3$
Thereofore,

$$f'(x)=4(x-2)^3(x+1)^3+3(x+1)^2(x-2)^4$$
$$=(x-2)^3(x+1)^2[4(x+1)+3(x-2)]$$
$$=(x-2)^3(x+1)^2(7x-2)$$

Now,

$$f'(x)=0$$
$$\Rightarrow x=-1, x=\frac{2}{7}, x=2$$

For values of x close to $\frac{2}{7}$ and to the left of $\frac{2}{7}, f'(x)>0$

Also, for values of x close to $\frac{2}{7}$ and to the right of $\frac{2}{7}, f'(x)<0$.

Thus, $x=\frac{2}{7}$ is the point of local maxima.

Now, for values of x close to 2 and to the left of $2, f'(x)<0$

Also, for values of x close to 2 and to the right of $2, f'(x)>0$.

Thus, $x=2$ is the point of local minima.

Now, as the value of x varies through -1, $f'(x)$ does not change its sign.

Thus, $x=-1$ is the point of inflexion.

Question 14:

Find the absolute maximum and minimum values of the function f given by $f(x)=\cos^2 x+\sin x, x\in[0,\pi]$.

Solution:

We have $f(x)=\cos^2 x+\sin x$

Therefore,

$$f'(x)=2\cos x(-\sin x)+\cos x$$
$$=-2\sin x\cos x+\cos x$$

Now,

$$f'(x)=0$$
$$\Rightarrow -2\sin x\cos x+\cos x=0$$
$$\Rightarrow \cos x=2\sin x\cos x$$
$$\Rightarrow \cos x(2\sin x-1)=0$$
$$\Rightarrow \sin x=\frac{1}{2} \text{ or } \cos x=0$$
$$\Rightarrow x=\frac{\pi}{6} \text{ or } \frac{\pi}{2} \qquad \because x\in[0,\pi]$$

Now, evaluating the value of f at critical points $x=\frac{\pi}{6},\frac{\pi}{2}$ and at the end points of the interval $[0,\pi]$ i.e., at $x=0$ and $x=\pi$, we have:

$$f\left(\frac{\pi}{6}\right)=\cos^2\left(\frac{\pi}{6}\right)+\sin\left(\frac{\pi}{6}\right)$$
$$=\left(\frac{\sqrt{3}}{2}\right)^2+\frac{1}{2}$$
$$=\frac{5}{4}$$
$$f(0)=\cos^2(0)+\sin(0)$$
$$=1+0$$
$$=1$$
$$f(\pi)=\cos^2(\pi)+\sin(\pi)$$
$$=(-1)^2+0$$
$$=1$$
$$f\left(\frac{\pi}{2}\right)=\cos^2\left(\frac{\pi}{2}\right)+\sin\left(\frac{\pi}{2}\right)$$
$$=0+1$$
$$=1$$

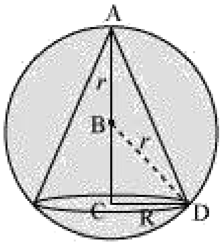
Hence, the absolute maximum value of f is $\frac{5}{4}$ occurring at $x=\frac{\pi}{6}$ and the absolute minimum value of f is 1 occurring at $x=0,\frac{\pi}{2},\pi$.

Question 15:

Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$.

Solution:

A sphere of fixed radius (r) is given.
Let R and h be the radius and the height of the cone respectively.



The volume (V) of the cone is given by

$$V=\frac{1}{3}\pi R^2 h$$

Now, from the right ΔBCD, we have:

$$BC=\sqrt{r^2-R^2}$$
$$\Rightarrow h=r+\sqrt{r^2-R^2}$$

Hence,

$$V=\frac{1}{3}\pi R^2\left(r+\sqrt{r^2-R^2}\right)$$
$$=\frac{1}{3}\pi R^2 r+\frac{1}{3}\pi R^2\sqrt{r^2-R^2}$$

Therefore,

$$\frac{dV}{dR}=\frac{2}{3}\pi Rr+\frac{2}{3}\pi R\sqrt{r^2-R^2}+\frac{\pi R^2}{3}.\frac{(-2R)}{2\sqrt{r^2-R^2}}$$
$$=\frac{2}{3}\pi Rr+\frac{2}{3}\pi R\sqrt{r^2-R^2}-\frac{\pi R^3}{3\sqrt{r^2-R^2}}$$
$$=\frac{2}{3}\pi Rr+\frac{2\pi R\left(r^2-R^2\right)-\pi R^3}{3\sqrt{r^2-R^2}}$$
$$=\frac{2}{3}\pi Rr+\frac{2\pi Rr^2-3\pi R^3}{3\sqrt{r^2-R^2}}$$

Now,

$$\frac{dV}{dR^2}=0$$
$$\Rightarrow\frac{2}{3}\pi Rr+\frac{2\pi Rr^2-3\pi R^3}{3\sqrt{r^2-R^2}}=0$$
$$\Rightarrow\frac{2\pi Rr}{3}=\frac{3\pi R^3-2\pi Rr^2}{3\sqrt{r^2-R^2}}$$
$$\Rightarrow 2r\sqrt{r^2-R^2}=3R^2-2r^2$$
$$\Rightarrow 4r^2\left(r^2-R^2\right)=\left(3R^2-2r^2\right)^2$$
$$\Rightarrow 4r^4-4r^2R^2=9R^4+4r^4-12R^2r^2$$
$$\Rightarrow 9R^4-8r^2R^2=0$$
$$\Rightarrow 9R^2=8r^2$$
$$\Rightarrow R^2=\frac{8r^2}{9}$$

Also,

$$\frac{d^2V}{dR^2}=\frac{2\pi r}{3}+\frac{3\sqrt{r^2-R^2}\left(2\pi r^2-9\pi R^2\right)-\left(2\pi Rr^2-3\pi R^3\right)(-6R)\frac{1}{2\sqrt{r^2-R^2}}}{9\left(r^2-R^2\right)}$$
$$=\frac{2\pi r}{3}+\frac{3\sqrt{r^2-R^2}\left(2\pi r^2-9\pi R^2\right)+\left(2\pi Rr^2-3\pi R^3\right)(3R)\frac{1}{\sqrt{r^2-R^2}}}{9\left(r^2-R^2\right)}$$

When, $R^2=\frac{8r^2}{9}$, $\Rightarrow\frac{d^2V}{dR^2}<0$.

Thus, the volume is the maximum when $R^2=\frac{8r^2}{9}$.

When, $R^2=\frac{8r^2}{9}$

Then, height of the cone

$$h = r + \sqrt{r^2 - \frac{8r^2}{9}}$$
$$= r + \sqrt{\frac{r^2}{9}}$$
$$= r + \frac{r}{3}$$
$$= \frac{4r}{3}$$

Hence, it can be seen that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius $\frac{4r}{3}$.

Question 16:

Let f be a function defined on $[a,b]$ such that $f'(x) > 0$, for all $x \in (a,b)$. Then prove that f is an increasing function on (a,b).

Solution:

Let $x_1, x_2 \in (a,b)$ such that $x_1 > x_2$

Consider the sub-interval $[x_1, x_2]$

Since $f(x)$ is differentiable on (a,b) and $[x_1, x_2] \subset (a,b)$.

Therefore, $f(x)$ is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2).

By the Lagrange's mean value theorem, there exists $c \in (x_1, x_2)$ such that

$$f(c) = \frac{f(x_2) - f(x_1)}{x_1 - x_2} \qquad \ldots(1)$$

Since, $f'(x) > 0$ for all $x \in (a,b)$, so in particular,

$$f'(c) > 0$$
$$\Rightarrow \frac{f(x_2) - f(x_1)}{x_1 - x_2} > 0 \qquad \text{[Using (1)]}$$
$$\Rightarrow f(x_2) - f(x_1) > 0$$
$$\Rightarrow f(x_2) > f(x_1)$$
$$\Rightarrow f(x_1) < f(x_2)$$

Since, x_1, x_2 are arbitrary points in (a,b).

Therefore, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in (a,b)$.

Hence, $f(x)$ is increasing on (a,b).

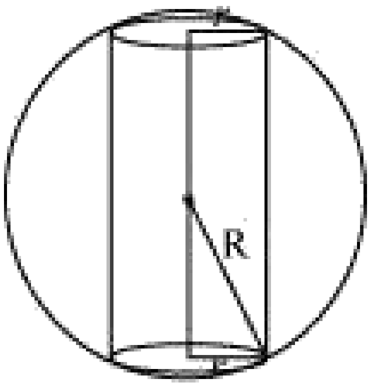
Question 17:

Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume.

Solution:

A sphere of fixed radius (R) is given.

Let r and h be the radius and the height of the cylinder respectively.



From the given figure, we have $h = 2\sqrt{R^2 - r^2}$

The volume (V) of the cylinder is given by,

$$V = \pi r^2 h$$
$$= 2\pi r^2 \sqrt{R^2 - r^2}$$

Therefore,

$$V = \pi r^2 h = 2\pi r^2 \sqrt{R^2 - r^2}$$
$$\frac{dV}{dr} = 4\pi r\sqrt{R^2 - r^2} + \frac{2\pi r^2 (-2r)}{2\sqrt{R^2 - r^2}}$$
$$= 4\pi r\sqrt{R^2 - r^2} - \frac{2\pi r^3}{\sqrt{R^2 - r^2}}$$
$$= \frac{4\pi r\sqrt{R^2 - r^2} - 2\pi r^3}{\sqrt{R^2 - r^2}}$$
$$= \frac{4\pi r R^2 - 6\pi r^3}{\sqrt{R^2 - r^2}}$$

Now,

$$\frac{dV}{dr} = 0$$
$$\Rightarrow \frac{4\pi r R^2 - 6\pi r^3}{\sqrt{R^2 - r^2}} = 0$$
$$\Rightarrow 4\pi r R^2 = 6\pi r^3$$
$$\Rightarrow r^2 = \frac{2R^2}{3}$$

Also,

$$\frac{d^2V}{dr^2} = \frac{\sqrt{R^2 - r^2}\left(4\pi R^2 - 18\pi r^2\right) - \left(4\pi r R^2 - 6\pi r^3\right)\frac{(-2r)}{2\sqrt{R^2 - r^2}}}{\left(R^2 - r^2\right)}$$
$$= \frac{\left(R^2 - r^2\right)\left(4\pi R^2 - 18\pi r^2\right) + r\left(4\pi r R^2 - 6\pi r^3\right)}{\left(R^2 - r^2\right)^{\frac{3}{2}}}$$
$$= \frac{\left(4\pi R^4 - 22\pi r^2 R^2 + 12\pi r^4 + 4\pi r^2 R^2\right)}{\left(R^2 - r^2\right)^{\frac{3}{2}}}$$

Now, it can be observed that at $r^2 = \frac{2R^2}{3}$, $\Rightarrow \frac{d^2V}{dr^2} < 0$

Thus, the volume is the maximum when $r^2 = \frac{2R^2}{3}$.

When, $r^2 = \frac{2R^2}{3}$

Then, the height of the cylinder is

$$2\sqrt{R^2 - \frac{2R^2}{3}} = 2\sqrt{\frac{R^2}{3}}$$
$$= \frac{2R}{\sqrt{3}}$$

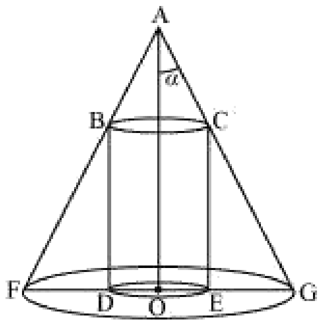
Hence, the volume of the cylinder is the maximum when the height of the cylinder is $\frac{2R}{\sqrt{3}}$.

Question 18:

Show that height of the cylinder of greatest volume which can be inscribed in a right circular cone of height h and semi vertical angle α is one-third that of the cone and the greatest volume of cylinder is $\frac{4}{27}\pi h^3 \tan^2 \alpha$.

Solution:

The given right circular cone of fixed height h and semi-vertical angle α can be drawn as:



Here, a cylinder of radius R and height H is inscribed in the cone.

Then, $\angle GAO = \alpha$, $OG = r$, $OA = h$, $OE = R$ and $CE = H$.

We have, $r = h\tan\alpha$

Now, since ΔAOG is similar to ΔCEG, we have:

$$\frac{AO}{OG}=\frac{CE}{EG}$$

$$\Rightarrow \frac{h}{r}=\frac{H}{r-R} \qquad [\because EG=OG-OE]$$

$$\Rightarrow H=\frac{h}{r}(r-R)$$

$$\Rightarrow H=\frac{h}{h\tan\alpha}(h\tan\alpha-R)$$

$$\Rightarrow H=\frac{1}{\tan\alpha}(h\tan\alpha-R)$$

Now, the volume (V) of the cylinder is given by,

$$V=\pi R^2 H$$

$$=\frac{\pi R^2}{\tan\alpha}(h\tan\alpha-R)$$

$$=\pi R^2 h-\frac{\pi R^3}{\tan\alpha}$$

Therefore,

$$\frac{dV}{dR}=2\pi Rh-\frac{3\pi R^2}{\tan\alpha}$$

Now,

$$\frac{dV}{dR}=0$$
$$\Rightarrow 2\pi Rh-\frac{3\pi R^2}{\tan\alpha}=0$$
$$\Rightarrow 2\pi Rh=\frac{3\pi R^2}{\tan\alpha}$$
$$\Rightarrow 2h\tan\alpha=3R$$
$$\Rightarrow R=\frac{2h}{3}\tan\alpha$$

Also

$$\frac{d^2V}{dR^2}=2\pi h-\frac{6\pi R}{\tan\alpha}$$

For $R=\frac{2h}{3}\tan\alpha$, we have:

$$\begin{aligned}\frac{d^2V}{dR^2}&=2\pi h-\frac{6\pi}{\tan\alpha}\left(\frac{2h}{3}\tan\alpha\right)\\&=2\pi h-4\pi h\\&=-2\pi h<0\end{aligned}$$

By second derivative test, the volume of the cylinder is the greatest when $R=\frac{2h}{3}\tan\alpha$.

When, $R=\frac{2h}{3}\tan\alpha$

Then,

$$\begin{aligned}H&=\frac{1}{\tan\alpha}\left(h\tan\alpha-\frac{2h}{3}\tan\alpha\right)\\&=\frac{1}{\tan\alpha}\left(\frac{h\tan\alpha}{3}\right)\\&=\frac{h}{3}\end{aligned}$$

Thus, the height of the cylinder is one-third the height of the cone when the volume of the cylinder is the greatest.

Thus, the maximum volume of the cylinder can be obtained as:

$$\begin{aligned}\pi\left(\frac{2h}{3}\tan\alpha\right)^2\left(\frac{h}{3}\right)&=\pi\left(\frac{4h^2}{9}\tan^2\alpha\right)\left(\frac{h}{3}\right)\\&=\frac{4}{27}\pi h^3\tan^2\alpha\end{aligned}$$

Hence, the given result is proved.

Question 19:

A cylindrical tank of radius 10 m is being filled with wheat at the rate of 314 cubic metre per hour. Then the depth of the wheat is increasing at the rate of

(A) $1m/h$ **(B)** $0.1m/h$ **(C)** $1.1m/h$ **(D)** $0.5m/h$

Solution:

Let r be the radius of the cylinder.

Then, volume (V) of the cylinder is given by,

$$V = \pi r^2 h$$
$$= \pi(10)^2 h$$
$$= 100\pi h$$

Differentiating with respect to time (t), we have:

$$\frac{dV}{dt} = 100\pi \frac{dh}{dt}$$

The tank is being filled with wheat at the rate of $314m^3/h$

$$\frac{dV}{dt} = 314m^3/h$$

Thus, we have:

$$100\pi \frac{dh}{dt} = 314$$
$$\frac{dh}{dt} = \frac{314}{100(3.14)}$$
$$= 1$$

Hence, the depth of wheat is increasing at the rate of $1m/h$.

Thus, the correct option is **A**.

Question 20:

The slope of the tangent to the curve $x = t^2 + 3t - 8$, $y = 2t^2 - 2t - 5$ at the point $(2,-1)$ is

(A) $\frac{22}{7}$ **(B)** $\frac{6}{7}$ **(C)** $\frac{7}{6}$ **(D)** $\frac{-6}{7}$

Solution:

The given curve is $x = t^2 + 3t - 8$ and $y = 2t^2 - 2t - 5$.

Therefore,

$$\frac{dx}{dt} = 2t + 3$$
$$\frac{dy}{dt} = 4t - 2$$

Hence,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4t-2}{2t+3}$$

The given point is $(2,-1)$

At $x = 2$, we have:

$$t^2 + 3t - 8 = 2$$
$$\Rightarrow t^2 + 3t - 10 = 0$$
$$\Rightarrow (t-2)(t+5) = 0$$
$$\Rightarrow t = 2 \text{ or } t = -5$$

At $y = -1$,we have:

$$2t^2 - 2t - 5 = -1$$
$$\Rightarrow 2t^2 - 2t - 4 = 0$$
$$\Rightarrow 2(t^2 - t - 2) = 0$$
$$\Rightarrow (t-2)(t+1) = 0$$
$$\Rightarrow t = 2 \text{ or } t = -1$$

The common value is $t = 2$

Hence, the slope of the tangent to the given curve at point $(2,-1)$ is

$$\left.\frac{dy}{dx}\right]_{t=2} = \frac{4(2)-2}{2(2)+3}$$
$$= \frac{8-2}{4+3}$$
$$= \frac{6}{7}$$

Thus, the correct option is **B**.

Question 21:

The line $y = mx + 1$ is a tangent to the curve $y^2 = 4x$ if the value of m is

(A) 1 **(B)** 2 **(C)** 3 **(D)** $\frac{1}{2}$

Solution:

The equation of the tangent to the given curve is $y = mx + 1$

Now, substituting $y = mx + 1$ in $y^2 = 4x$, we get:

$$(mx+1)^2 = 4x$$
$$\Rightarrow m^2x^2 + 1 + 2mx - 4x = 0$$
$$\Rightarrow m^2x^2 + (2m-4)x + 1 = 0 \qquad \ldots(1)$$

Since a tangent touches the curve at one point, the roots of equation (1) must be equal.

Therefore, we have:

$$\text{Discriminant} = 0$$
$$\Rightarrow (2m-4)^2 - 4(m^2)(1) = 0$$
$$\Rightarrow 4m^2 + 16 - 16m - 4m^2 = 0$$
$$\Rightarrow 16 - 16m = 0$$
$$\Rightarrow m = 1$$

Hence, the required value of m is 1.

Thus, the correct option is **A**.

Question 22:

The normal at the point $(1,1)$ on the curve $2y + x^2 = 3$ is

(A) $x + y = 0$ **(B)** $x - y = 0$ **(C)** $x + y + 1 = 0$ **(D)** $x - y = 1$

Solution:

The equation of the given curve is $2y + x^2 = 3$

Differentiating with respect to x, we have:

$$\frac{2dy}{dx}+2x=0$$
$$\Rightarrow \frac{dy}{dx}=-x$$
$$\Rightarrow \left.\frac{dy}{dx}\right]_{(1,1)}=-1$$

The slope of the normal to the given curve at point $(1,1)$ is

$$\frac{-1}{\left.\frac{dy}{dx}\right]_{(1,1)}}=1$$

Hence, the equation of the normal to the given curve at $(1,1)$ is given as:

$$\Rightarrow y-1=1(x-1)$$
$$\Rightarrow y-1=x-1$$
$$\Rightarrow x-y=0$$

Thus, the correct option is **B**.

Question 23:

The normal to the curve $x^2=4y$ passing $(1,2)$ is

(A) $x+y=3$ **(B)** $x-y=3$ **(C)** $x+y=1$ **(D)** $x-y=1$

Solution:

The equation of the given curve is $x^2=4y$

Differentiating with respect to x, we have:

$$2x=4.\frac{dy}{dx}$$
$$\Rightarrow \frac{dy}{dx}=\frac{x}{2}$$

The slope of the normal to the given curve at point (h,k) is

$$\frac{-1}{\left.\frac{dy}{dx}\right]_{(h,k)}}=-\frac{2}{h}$$

Hence, the equation of the normal to the given curve at (h,k) is given as:

$$y-k=-\frac{2}{h}(x-h)$$

Now, it is given that the normal passes through the point $(1,2)$

Therefore, we have:

$$2-k=-\frac{2}{h}(1-h)$$
$$\Rightarrow k=2+\frac{2}{h}(1-h) \qquad \ldots(1)$$

Since (h,k) lies on the curve $x^2=4y$, we have $h^2=4k$

$$\Rightarrow k=\frac{h^2}{4}$$

From equation (1), we have:

$$\frac{h^2}{4}=2+\frac{2}{h}(1-h)$$
$$\Rightarrow \frac{h^3}{4}=2h+2-2h$$
$$\Rightarrow \frac{h^3}{4}=2$$
$$\Rightarrow h^3=8$$
$$\Rightarrow h=2$$

Therefore,

$$k=\frac{h^2}{4}$$
$$\Rightarrow k=1$$

Hence, the equation of the normal is given as:

$$\Rightarrow y-1=\frac{-2}{2}(x-2)$$
$$\Rightarrow y-1=-x+2$$
$$\Rightarrow x+y=3$$

Thus, the correct option is **A**.

Question 24:

The points on the curve $9y^2 = x^3$, where the normal to the curve makes equal intercepts with the axes are

(A) $\left(4, \pm\frac{8}{3}\right)$ **(B)** $4, \frac{-8}{3}$ **(C)** $\left(4, \pm\frac{3}{8}\right)$ **(D)** $\left(\pm 4, \frac{3}{8}\right)$

Solution:

The equation of the given curve is $9y^2 = x^3$

Differentiating with respect to x, we have:

$$9(2y)\frac{dy}{dx} = 3x^2$$
$$\Rightarrow \frac{dy}{dx} = \frac{x^2}{6y}$$

The slope of the normal to the given curve at point (x_1, y_1) is

$$\frac{-1}{\left.\frac{dy}{dx}\right]_{(x_1, y_1)}} = -\frac{6y_1}{x_1^2}$$

The equation of the normal to the curve at (x_1, y_1) is

$$y - y_1 = -\frac{6y_1}{x_1^2}(x - x_1)$$
$$\Rightarrow x_1^2 y - x_1^2 y_1 = -6xy_1 + 6x_1y_1$$
$$\Rightarrow 6xy_1 + x_1^2 y = 6x_1y_1 + x_1^2 y_1$$
$$\Rightarrow \frac{6xy_1 + x_1^2 y}{6x_1y_1 + x_1^2 y_1} = 1$$
$$\Rightarrow \frac{6xy_1}{6x_1y_1 + x_1^2 y_1} + \frac{x_1^2 y}{6x_1y_1 + x_1^2 y_1} = 1$$
$$\Rightarrow \frac{x}{\frac{x_1(6 + x_1)}{6}} + \frac{y}{\frac{y_1(6 + x_1)}{x_1}} = 1$$

It is given that the normal makes equal intercepts with the axes.

Therefore, we have:

$$\Rightarrow \frac{x_1(6+x_1)}{6} = \frac{y_1(6+x_1)}{x_1}$$

$$\Rightarrow \frac{x_1}{6} = \frac{y_1}{x_1}$$

$$\Rightarrow x_1^2 = 6y_1 \qquad \ldots(1)$$

Also, the point (x_1, y_1) lies on the curve, so we have

$$9y_1^2 = x_1^3 \qquad \ldots(2)$$

From (1) and (2), we have:

$$\Rightarrow 9\left(\frac{x_1^2}{6}\right)^2 = x_1^3$$

$$\Rightarrow \frac{x_1^4}{4} = x_1^3$$

$$\Rightarrow x_1 = 4$$

From (2), we have:

$$\Rightarrow 9y_1^2 = 4^3$$

$$\Rightarrow y_1^2 = \frac{64}{9}$$

$$\Rightarrow y_1 = \pm\frac{8}{3}$$

Hence, the required points are $\left(4, \pm\frac{8}{3}\right)$

Thus, the correct option is **A**.