Chapter 2 Inverse Trigonometric Functions

EXERCISE 2.1

Question 1:

Find the principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$.

Solution:

Let, $\sin^{-1}\left(-\frac{1}{2}\right) = y$

Hence,

$$\begin{aligned}\sin y &= \left(-\frac{1}{2}\right) \\ &= -\sin\left(\frac{\pi}{6}\right) \\ &= \sin\left(-\frac{\pi}{6}\right)\end{aligned}$$

Range of the principal value of $\sin^{-1}(x)$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Thus, principal value of $\sin^{-1}\left(-\frac{1}{2}\right) = \left(-\frac{\pi}{6}\right)$.

Question 2:

Find the principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$.

Solution:

Let, $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = y$

Hence,

$$\begin{aligned}\cos y &= \left(\frac{\sqrt{3}}{2}\right) \\ &= \cos\frac{\pi}{6}\end{aligned}$$

Range of the principal value of $\cos^{-1}(x)$ is $(0, \pi)$.

Thus, principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \left(\frac{\pi}{6}\right)$

Question 3:

Find the principal value of $\operatorname{cosec}^{-1}(2)$.

Solution:

Let, $\operatorname{cosec}^{-1}(2) = y$

Hence,

$$\operatorname{cosec} y = 2$$

$$= \operatorname{cosec}\left(\frac{\pi}{6}\right)$$

Range of the principal value of $\operatorname{cosec}^{-1}(x) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

Thus, principal value of $\operatorname{cosec}^{-1}(2) = \left(\frac{\pi}{6}\right)$.

Question 4:

Find the principal value of $\tan^{-1}\left(-\sqrt{3}\right)$

Solution:

Let, $\tan^{-1}\left(-\sqrt{3}\right) = y$

Hence,

$$\tan y = -\sqrt{3}$$

$$= -\tan\left(\frac{\pi}{3}\right)$$

$$= \tan\left(-\frac{\pi}{3}\right)$$

Range of the principal value of $\tan^{-1}(x) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Thus, principal value of $\tan^{-1}\left(-\sqrt{3}\right) = \left(-\frac{\pi}{3}\right)$.

Question 5:

Find the principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$

Solution:

Let, $\cos^{-1}\left(-\frac{1}{2}\right)=y$

Hence,

$$\begin{aligned}\cos y&=-\frac{1}{2}\\&=-\cos\left(\frac{\pi}{3}\right)\\&=\cos\left(\pi-\frac{\pi}{3}\right)\\&=\cos\left(\frac{2\pi}{3}\right)\end{aligned}$$

Range of the principal value of $\cos^{-1}(x)=[0,\pi]$

Thus, principal value of $\cos^{-1}\left(-\frac{1}{2}\right)=\left(\frac{2\pi}{3}\right)$.

Question 6:

Find the principal value of $\tan^{-1}(-1)$

Solution:

Let, $\tan^{-1}(-1)=y$

Hence,

$$\begin{aligned}\tan\ y&=-1\\&=-\tan\left(\frac{\pi}{4}\right)\\&=\tan\left(-\frac{\pi}{4}\right)\end{aligned}$$

Range of the principal value of $\tan^{-1}(x)=\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

Thus, principal value of $\tan^{-1}(-1)=\left(-\frac{\pi}{4}\right)$.

Question 7:

Find the principal value of $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

Solution:

Let, $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)=y$

Hence,

$$\sec y=\frac{2}{\sqrt{3}}$$
$$=\sec\left(\frac{\pi}{6}\right)$$

Range of the principal value of $\sec^{-1}(x)=[0,\pi]-\left\{\frac{\pi}{2}\right\}$

Thus, principal value of $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)=\left(\frac{\pi}{6}\right)$.

Question 8:

Find the principal value of $\cot^{-1}\left(\sqrt{3}\right)$

Solution:

Let, $\cot^{-1}\left(\sqrt{3}\right)=y$

Hence,

$$\cot y=\sqrt{3}$$
$$=\cot\left(\frac{\pi}{6}\right)$$

Range of the principal value of $\cot^{-1}(x)=(0,\pi)$

Thus, principal value of $\cot^{-1}\left(\sqrt{3}\right)=\left(\frac{\pi}{6}\right)$.

Question 9:

Find the principal value of $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

Solution:

Let, $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)=y$

Hence,

$$\cos y = -\frac{1}{\sqrt{2}}$$
$$= -\cos\left(\frac{\pi}{4}\right)$$
$$= \cos\left(-\frac{\pi}{4}\right)$$
$$= \cos\left(\pi - \frac{\pi}{4}\right)$$
$$= \cos\left(\frac{3\pi}{4}\right)$$

Range of the principal value of $\cos^{-1}(x) = [0, \pi]$

Thus, principal value of $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \left(\frac{3\pi}{4}\right)$.

Question 10:

Find the principal value of $\operatorname{cosec}^{-1}\left(-\sqrt{2}\right)$

Solution:

Let, $\operatorname{cosec}^{-1}\left(-\sqrt{2}\right) = y$

Hence,

$$\operatorname{cosec} y = -\sqrt{2}$$
$$= -\operatorname{cosec}\left(\frac{\pi}{4}\right)$$
$$= \operatorname{cosec}\left(-\frac{\pi}{4}\right)$$

Range of the principal value of $\operatorname{cosec}^{-1}(x) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

Thus, principal value of $\operatorname{cosec}^{-1}\left(-\sqrt{2}\right) = \left(-\frac{\pi}{4}\right)$.

Question 11:

Find the value of $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$.

Solution:

Let, $\tan^{-1}(1)=x$

Hence,

$$\tan x=1$$
$$=\tan\left(\frac{\pi}{4}\right)$$

Therefore,

$$\tan^{-1}(1)=\left(\frac{\pi}{4}\right)$$

Now, let $\cos^{-1}\left(-\frac{1}{2}\right)=y$

Hence,

$$\cos y=-\frac{1}{2}$$
$$=-\cos\left(\frac{\pi}{3}\right)$$
$$=\cos\left(\pi-\frac{\pi}{3}\right)$$
$$=\cos\left(\frac{2\pi}{3}\right)$$

Therefore,

$$\cos^{-1}\left(-\frac{1}{2}\right)=\frac{2\pi}{3}$$

Again, let $\sin^{-1}\left(-\frac{1}{2}\right)=z$

Hence,

$$\sin z=-\frac{1}{2}$$
$$=-\sin\left(\frac{\pi}{6}\right)$$
$$=\sin\left(-\frac{\pi}{6}\right)$$

Therefore,

$$\sin^{-1}\left(-\frac{1}{2}\right)=-\frac{\pi}{6}$$

Thus,

$$\tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right)+\sin^{-1}\left(-\frac{1}{2}\right)=\frac{\pi}{4}+\frac{2\pi}{3}-\frac{\pi}{6}$$
$$=\frac{3\pi+8\pi-2\pi}{12}$$
$$=\frac{9\pi}{12}$$
$$=\frac{3\pi}{4}$$

Question 12:

Find the value of $\cos^{-1}\left(\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)$

Solution:

Let, $\tan^{-1}(1)=x$

Hence,

$$\cos x=\frac{1}{2}$$
$$=\cos\left(\frac{\pi}{3}\right)$$

Therefore,

$$\cos^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3}$$

Let, $\sin^{-1}\left(\frac{1}{2}\right)=y$

Hence,

$$\sin y=\frac{1}{2}$$
$$=\sin\left(\frac{\pi}{6}\right)$$

Therefore,

$$\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}$$

Thus

$$\cos^{-1}\left(\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3}+2\left(\frac{\pi}{6}\right)$$
$$=\frac{2\pi}{3}$$

Question 13:

Find the value of $\sin^{-1}x=y$, then

(A) $0\leq y\leq\pi$ (B) $-\frac{\pi}{2}\leq y\leq\frac{\pi}{2}$

(C) $0\leq y\leq\pi$ (D) $-\frac{\pi}{2}<y<\frac{\pi}{2}$

Solution:

It is given that $\sin^{-1}x=y$

Range of the principal value of $\sin^{-1}x=\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

Thus, $-\frac{\pi}{2}\leq y\leq\frac{\pi}{2}$

The answer is B.

Question 14:

Find the value of $\tan^{-1}\sqrt{3}-\sec^{-1}(-2)$ is equal to

(A) 0 (B) $-\frac{\pi}{3}$

(C) $\frac{\pi}{3}$ (D) $\frac{2\pi}{3}$

Solution:

Let $\tan^{-1}\left(\sqrt{3}\right)=x$

Hence,

$$\tan x=\sqrt{3}$$
$$=\tan\left(\frac{\pi}{3}\right)$$

Range of the principal value of $\tan^{-1}x=\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

Therefore, $\tan^{-1}\left(\sqrt{3}\right)=\left(\frac{\pi}{3}\right)$

Let $\sec^{-1}(-2)=y$

Hence,

$$\begin{aligned}\sec y&=(-2)\\&=-\sec\left(\frac{\pi}{3}\right)\\&=\sec\left(-\frac{\pi}{3}\right)\\&=\sec\left(\pi-\frac{\pi}{3}\right)\\&=\sec\left(\frac{2\pi}{3}\right)\end{aligned}$$

Range of the principal value of $\sec^{-1}\text{x}=[0,\pi]-\left\{\frac{\pi}{2}\right\}$

Therefore, $\sec^{-1}(-2)=\frac{2\pi}{3}$

Thus,

$$\begin{aligned}\tan^{-1}\sqrt{3}-\sec^{-1}(-2)&=\frac{\pi}{3}-\frac{2\pi}{3}\\&=-\frac{\pi}{3}\end{aligned}$$

The answer is B.

EXERCISE 2.2

Question 1:

Prove $3\sin^{-1}x = \sin^{-1}\left(3x - 4x^3\right), x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$.

Solution:

Let $x = \sin\theta$

Hence, $\sin^{-1}(x) = \theta$

Now,

$$\begin{aligned}
RHS &= \sin^{-1}\left(3x - 4x^3\right) \\
&= \sin^{-1}\left(3\sin\theta - 4\sin^3\theta\right) \\
&= \sin^{-1}\left(\sin 3\theta\right) \\
&= 3\theta \\
&= 3\sin^{-1}x \\
&= LHS
\end{aligned}$$

Question 2:

Prove $3\cos^{-1}x = \cos^{-1}\left(4x^3 - 3x\right), x \in \left[\frac{1}{2}, 1\right]$.

Solution:

Let $x = \cos\theta$

Hence, $\cos^{-1}(x) = \theta$

Now,

$$\begin{aligned}
RHS &= \cos^{-1}\left(4x^3 - 3x\right) \\
&= \cos^{-1}\left(4\cos^3\theta - 3\cos\theta\right) \\
&= \cos^{-1}\left(\cos 3\theta\right) \\
&= 3\theta \\
&= 3\cos^{-1}x \\
&= LHS
\end{aligned}$$

Question 3:

Prove $\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\frac{1}{2}$.

Solution:

Since we know that $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}$

Now,

$$\begin{aligned}
LHS &= \tan^{-1}\frac{2}{11}+\tan^{-1}\frac{7}{24} \\
&= \tan^{-1}\frac{\frac{2}{11}+\frac{7}{24}}{1-\frac{2}{11}\cdot\frac{7}{24}} \\
&= \tan^{-1}\left(\frac{\frac{48+77}{264}}{\frac{264-14}{264}}\right) \\
&= \tan^{-1}\left(\frac{125}{250}\right) \\
&= \tan^{-1}\left(\frac{1}{2}\right) \\
&= RHS
\end{aligned}$$

Question 4:

Prove $2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{31}{17}$.

Solution:

Since we know that $2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}$ and $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}$

Now,

$$
\begin{aligned}
LHS &= 2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7} \\
&= \tan^{-1}\frac{2\times\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2}+\tan^{-1}\frac{1}{7} \\
&= \tan^{-1}\left(\frac{4}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right) \\
&= \tan^{-1}\left(\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}\cdot\frac{1}{7}}\right) \\
&= \tan^{-1}\left(\frac{\frac{28+3}{21}}{\frac{21-4}{21}}\right) \\
&= \tan^{-1}\left(\frac{31}{17}\right) \\
&= RHS
\end{aligned}
$$

Question 5:

Write the function in the simplest form: $\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}, x \neq 0$

Solution:

Let $x = \tan\theta \Rightarrow \theta = \tan^{-1} x$

Hence,

$$\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}=\tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$$=\tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)$$

$$=\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$=\tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$$

$$=\tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$=\frac{\theta}{2}$$

$$=\frac{1}{2}\tan^{-1}x$$

Question 6:

Write the function in the simplest form: $\tan^{-1}\frac{1}{\sqrt{x^2-1}},|x|>1$

Solution:

Let $x=\operatorname{cosec}\theta \Rightarrow \theta=\operatorname{cosec}^{-1}x$

Hence,

$$\tan^{-1}\frac{1}{\sqrt{x^2-1}}=\tan^{-1}\frac{1}{\sqrt{\operatorname{cosec}^2\theta-1}}$$

$$=\tan^{-1}\left(\frac{1}{\cot\theta}\right)$$

$$=\tan^{-1}(\tan\theta)$$

$$=\theta$$

$$=\operatorname{cosec}^{-1}x$$

$$=\frac{\pi}{2}-\sec^{-1}x$$

Question 7:

Write the function in the simplest form: $\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right),0<x<\pi$

Solution:

Since, $1-\cos x=2\sin^2\frac{x}{2}$ and $1+\cos x=2\cos^2\frac{x}{2}$

Hence,

$$\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)=\tan^{-1}\left(\sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}}\right)$$
$$=\tan^{-1}\left(\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right)$$
$$=\tan^{-1}\left(\tan\frac{x}{2}\right)$$
$$=\frac{x}{2}$$

Question 8:

Write the function in the simplest form: $\tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right), 0<x<\pi$

Solution:

$$\tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)=\tan^{-1}\left(\frac{\frac{\cos x-\sin x}{\cos x}}{\frac{\cos x+\sin x}{\cos x}}\right)$$
$$=\tan^{-1}\left(\frac{1-\frac{\sin x}{\cos x}}{1+\frac{\sin x}{\cos x}}\right)$$
$$=\tan^{-1}\left(\frac{1-\tan x}{1+\tan x}\right)$$
$$=\tan^{-1}(1)-\tan^{-1}(\tan x)$$
$$=\frac{\pi}{4}-x$$

Question 9:

Write the function in the simplest form: $\tan^{-1}\frac{x}{\sqrt{a^2-x^2}}, |x|<a$

Solution:

Let $x = a\sin\theta \Rightarrow \theta = \sin^{-1}\left(\frac{x}{a}\right)$

Hence,

$$\begin{aligned}\tan^{-1}\frac{x}{\sqrt{a^2-x^2}} &= \tan^{-1}\left(\frac{a\sin\theta}{\sqrt{a^2-a^2\sin^2\theta}}\right)\\ &= \tan^{-1}\left(\frac{a\sin\theta}{a\sqrt{1-\sin^2\theta}}\right)\\ &= \tan^{-1}\left(\frac{a\sin\theta}{a\cos\theta}\right)\\ &= \tan^{-1}(\tan\theta)\\ &= \theta\\ &= \sin^{-1}\frac{x}{a}\end{aligned}$$

Question 10:

Write the function in the simplest form: $\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right), a>0; \frac{-a}{\sqrt{3}} \le x \le \frac{a}{\sqrt{3}}$

Solution:

Let $x = a\tan\theta \Rightarrow \theta = \tan^{-1}\left(\frac{x}{a}\right)$

Hence,

$$\begin{aligned}\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right) &= \tan^{-1}\left(\frac{3a^2\cdot a\tan\theta - a^3\tan^3\theta}{a^3-3a.a^2\tan^2\theta}\right)\\ &= \tan^{-1}\left(\frac{3a^3\tan\theta - a^3\tan^3\theta}{a^3-3a^3\tan^2\theta}\right)\\ &= \tan^{-1}(\tan 3\theta)\\ &= 3\theta\\ &= 3\tan^{-1}\frac{x}{a}\end{aligned}$$

Question 11:

Write the function in the simplest form: $\tan^{-1}\left[2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right]$

Solution:

Let $\sin^{-1}\frac{1}{2}=x$

Hence,

$$\sin x=\frac{1}{2}$$
$$=\sin\left(\frac{\pi}{6}\right)$$
$$x=\frac{\pi}{6}$$
$$\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}$$

Therefore,

$$\tan^{-1}\left[2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right]=\tan^{-1}\left[2\cos\left(2\times\frac{\pi}{6}\right)\right]$$
$$=\tan^{-1}\left[2\cos\frac{\pi}{3}\right]$$
$$=\tan^{-1}\left[2\times\frac{1}{2}\right]$$
$$=\tan^{-1}[1]$$
$$=\frac{\pi}{4}$$

Question 12:

Find the value of $\cot\left(\tan^{-1}a+\cot^{-1}a\right)$

Solution:

Since $\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}$

Hence,

$$\cot\left(\tan^{-1}a+\cot^{-1}a\right)=\cot\left(\frac{\pi}{2}\right)$$
$$=0$$

Question 13:

Find the value of $\tan\frac{1}{2}\left(\sin^{-1}\frac{2x}{1+x^2}+\cos^{-1}\frac{1-y^2}{1+y^2}\right), |x|<1, y>0$ and $xy<1$.

Solution:

Let $x=\tan\theta \Rightarrow \theta=\tan^{-1}x$

Hence,

$$\begin{aligned}\sin^{-1}\frac{2x}{1+x^2} &= \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\\ &= \sin^{-1}(\sin 2\theta)\\ &= 2\theta\\ &= 2\tan^{-1}x\end{aligned}$$

Now, let $y=\tan\phi \Rightarrow \phi=\tan^{-1}y$

Hence,

$$\begin{aligned}\cos^{-1}\frac{1-y^2}{1+y^2} &= \cos^{-1}\left(\frac{1-\tan^2\phi}{1+\tan^2\phi}\right)\\ &= \cos^{-1}(\cos 2\phi)\\ &= 2\phi\\ &= 2\tan^{-1}y\end{aligned}$$

Therefore,

$$\begin{aligned}\tan\frac{1}{2}\left(\sin^{-1}\frac{2x}{1+x^2}+\cos^{-1}\frac{1-y^2}{1+y^2}\right) &= \tan\frac{1}{2}\left(2\tan^{-1}x+2\tan^{-1}y\right)\\ &= \tan\left(\tan^{-1}x+\tan^{-1}y\right)\\ &= \tan\left[\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]\\ &= \left(\frac{x+y}{1-xy}\right)\end{aligned}$$

Question 14:

If $\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$, find the value of x.

Solution:

It is given that $\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$

Since we know that $\sin(x+y)=\sin x\cos y+\cos x\sin y$
Therefore,

$$\sin\left(\sin^{-1}\frac{1}{5}\right)\cos\left(\cos^{-1}x\right)+\cos\left(\sin^{-1}\frac{1}{5}\right)\sin\left(\cos^{-1}x\right)=1$$

$$\left(\frac{1}{5}\right)\times(x)+\cos\left(\sin^{-1}\frac{1}{5}\right)\sin\left(\cos^{-1}x\right)=1$$

$$\frac{x}{5}+\cos\left(\sin^{-1}\frac{1}{5}\right)\sin\left(\cos^{-1}x\right)=1 \qquad \ldots(1)$$

Now, let $\sin^{-1}\frac{1}{5}=y \Rightarrow \sin y=\frac{1}{5}$

Then,

$$\cos y=\sqrt{1-\left(\frac{1}{5}\right)^2}$$

$$=\frac{2\sqrt{6}}{5}$$

$$y=\cos^{-1}\left(\frac{2\sqrt{6}}{5}\right)$$

Therefore,

$$\sin^{-1}\frac{1}{5}=\cos^{-1}\left(\frac{2\sqrt{6}}{5}\right) \qquad \ldots(2)$$

Now, let $\cos^{-1}x=z \Rightarrow \cos z=x$
Then,

$$\sin z=\sqrt{1-x^2}$$

$$z=\sin^{-1}\sqrt{1-x^2}$$

Therefore,

$$\cos^{-1}x=\sin^{-1}\sqrt{1-x^2} \qquad \ldots(3)$$

From $(1),(2)$ and (3), we have

$$\Rightarrow \frac{x}{5} + \cos\left(\cos^{-1}\frac{2\sqrt{6}}{5}\right)\sin\left(\sin^{-1}\sqrt{1-x^2}\right) = 1$$

$$\Rightarrow \frac{x}{5} + \frac{2\sqrt{6}}{5}\sqrt{1-x^2} = 1$$

$$\Rightarrow x + 2\sqrt{6}\sqrt{1-x^2} = 5$$

$$\Rightarrow 5 - x = 2\sqrt{6}\sqrt{1-x^2}$$

On squaring both the sides

$$25 + x^2 - 10x = 24 - 24x^2$$

$$25x^2 - 10x + 1 = 0$$

$$(5x-1)^2 = 0$$

$$(5x-1) = 0$$

$$x = \frac{1}{5}$$

Question 15:

If $\tan^{-1}\frac{x-1}{x-2} + \tan^{-1}\frac{x+1}{x+2} = \frac{\pi}{4}$, find the value of x.

Solution:

It is given that $\tan^{-1}\frac{x-1}{x-2} + \tan^{-1}\frac{x+1}{x+2} = \frac{\pi}{4}$

Since $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

Therefore,

$$\Rightarrow \tan^{-1}\left(\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)}\right)=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left[\frac{(x-1)(x+2)+(x+1)(x-2)}{(x+2)(x-2)-(x-1)(x+1)}\right]=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left[\frac{x^2+x-2+x^2-x-2}{x^2-4-x^2+1}\right]=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left[\frac{2x^2-4}{-3}\right]=\frac{\pi}{4}$$

$$\Rightarrow \tan\left[\tan^{-1}\frac{4-2x^2}{3}\right]=\tan\frac{\pi}{4}$$

$$\Rightarrow \frac{4-2x^2}{3}=1$$

$$\Rightarrow 4-2x^2=3$$

$$\Rightarrow 2x^2=1$$

$$\Rightarrow x^2=\frac{1}{2}$$

$$\Rightarrow x=\pm\frac{1}{\sqrt{2}}$$

Question 16:

Find the value of $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$.

Solution:

Since, $\sin\theta = \sin(\pi-\theta)$

Therefore,

$$\sin^{-1}\left(\sin\frac{2\pi}{3}\right)=\sin^{-1}\left[\sin\left(\pi-\frac{2\pi}{3}\right)\right]$$

$$=\sin^{-1}\left(\sin\frac{\pi}{3}\right)$$

$$=\frac{\pi}{3}$$

Question 17:

Find the value of $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$.

Solution:

Since, $\tan\theta = -\tan(\pi - \theta)$

Therefore,

$$\begin{aligned}
\tan^{-1}\left(\tan\frac{3\pi}{4}\right) &= \tan^{-1}\left[-\tan\left(-\frac{3\pi}{4}\right)\right] \\
&= \tan^{-1}\left[-\tan\left(\pi - \frac{\pi}{4}\right)\right] \\
&= \tan^{-1}\left[-\tan\left(\frac{\pi}{4}\right)\right] \\
&= \tan^{-1}\left[\tan\left(-\frac{\pi}{4}\right)\right] \\
&= \left(-\frac{\pi}{4}\right)
\end{aligned}$$

Question 18:

Find the value of $\tan\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right)$.

Solution:

Let $\sin^{-1}\frac{3}{5} = x \Rightarrow \sin x = \frac{3}{5}$

Then,

$$\Rightarrow \cos x = \sqrt{1 - \sin^2 x} = \frac{4}{5}$$

$$\Rightarrow \sec x = \frac{5}{4}$$

Therefore,

$$\begin{aligned}
\tan x &= \sqrt{\sec^2 x - 1} \\
&= \sqrt{\frac{25}{16} - 1} \\
&= \frac{3}{4}
\end{aligned}$$

$$x = \tan^{-1}\frac{3}{4}$$

$$\sin^{-1}\frac{3}{5} = \tan^{-1}\frac{3}{4} \qquad \ldots(1)$$

Now,

$$\cot^{-1}\frac{3}{2}=\tan^{-1}\frac{2}{3} \qquad \ldots(2)$$

Thus, by using (1) and (2)

$$\begin{aligned}\tan\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{2}{3}\right)&=\tan\left(\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{2}{3}\right)\\&=\tan\left[\tan^{-1}\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\cdot\frac{2}{3}}\right]\\&=\tan\left(\tan^{-1}\frac{17}{6}\right)\\&=\frac{17}{6}\end{aligned}$$

Question 19:

$\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ is equal to

(A) $\frac{7\pi}{6}$ (B) $\frac{5\pi}{6}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{6}$

Solution:

$$\begin{aligned}\cos^{-1}\left(\cos\frac{7\pi}{6}\right)&=\cos^{-1}\left(\cos\frac{-7\pi}{6}\right)\\&=\cos^{-1}\left[\cos\left(2\pi-\frac{7\pi}{6}\right)\right]\\&=\cos^{-1}\left[\cos\left(\frac{5\pi}{6}\right)\right]\\&=\frac{5\pi}{6}\end{aligned}$$

Thus, the correct option is B.

Question 20:

$\sin\left(\frac{\pi}{3}-\sin^{-1}\left(-\frac{1}{2}\right)\right)$ is equal to

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{1}{4}$ (D) 1

Solution:

Let $\sin^{-1}\left(-\frac{1}{2}\right)=x$

Hence,

$$\begin{aligned}\sin x &= -\frac{1}{2}\\ &= -\sin\frac{\pi}{6}\\ &= \sin\left(-\frac{\pi}{6}\right)\\ x &= -\frac{\pi}{6}\end{aligned}$$

Since, Range of principal value of $\sin^{-1}(x)=\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$.
Therefore,

$$\sin^{-1}\left(-\frac{1}{2}\right)=-\frac{\pi}{6}$$

Then,

$$\begin{aligned}\sin\left(\frac{\pi}{3}-\sin^{-1}\left(-\frac{1}{2}\right)\right) &= \sin\left(\frac{\pi}{3}+\frac{\pi}{6}\right)\\ &= \sin\left(\frac{\pi}{2}\right)\\ &= 1\end{aligned}$$

Thus, the correct option is D.

Question 21:

Find the values of $\tan^{-1}\sqrt{3}-\cot^{-1}\left(-\sqrt{3}\right)$ is equal to

(A) π (B) $-\frac{\pi}{2}$ (C) 0 (D) $2\sqrt{3}$

Solution:

Let $\tan^{-1}\sqrt{3}=x$

Hence,

$$\tan x = \sqrt{3} = \tan \frac{\pi}{3}, \text{ where } \frac{\pi}{3} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Therefore, $\tan^{-1} \sqrt{3} = \frac{\pi}{3}$

Now, let $\cot^{-1}\left(-\sqrt{3}\right) = y$

Hence,

$$\begin{aligned} \cot y &= \left(-\sqrt{3}\right) \\ &= -\cot\left(\frac{\pi}{6}\right) \\ &= \cot\left(\pi - \frac{\pi}{6}\right) \\ &= \cot\left(\frac{5\pi}{6}\right) \end{aligned}$$

Since, Range of principal value of $\cot^{-1} x = (0, \pi)$

Therefore,

$$\cot^{-1}\left(-\sqrt{3}\right) = \frac{5\pi}{6}$$

Then,

$$\begin{aligned} \tan^{-1} \sqrt{3} - \cot^{-1}\left(-\sqrt{3}\right) &= \frac{\pi}{3} - \frac{5\pi}{6} \\ &= -\frac{\pi}{2} \end{aligned}$$

Thus, the correct option is B.

MISCELLANEOUS EXERCISE

Question 1:

Find the value of $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$.

Solution:

$$\cos^{-1}\left(\cos\frac{13\pi}{6}\right)=\cos^{-1}\left[\cos\left(2\pi+\frac{\pi}{6}\right)\right]$$
$$=\cos^{-1}\left[\cos\frac{\pi}{6}\right]$$
$$=\frac{\pi}{6}$$

Question 2:

Find the value of $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$.

Solution:

$$\tan^{-1}\left(\tan\frac{7\pi}{6}\right)=\tan^{-1}\left[\tan\left(2\pi-\frac{5\pi}{6}\right)\right]$$
$$=\tan^{-1}\left[-\tan\left(\frac{5\pi}{6}\right)\right]$$
$$=\tan^{-1}\left[\tan\left(\pi-\frac{5\pi}{6}\right)\right]$$
$$=\tan^{-1}\left[\tan\frac{\pi}{6}\right]$$
$$=\frac{\pi}{6}$$

Question 3:

Prove that $2\sin^{-1}\frac{3}{5}=\tan^{-1}\frac{24}{7}$.

Solution:

Let $\sin^{-1}\frac{3}{5}=x\Rightarrow\sin x=\frac{3}{5}$

Then,

$$\cos x=\sqrt{1-\left(\frac{3}{5}\right)^2}=\frac{4}{5}$$

Therefore,

$$\tan x=\frac{3}{4}$$

$$x=\tan^{-1}\frac{3}{4}$$

$$\sin^{-1}\frac{3}{5}=\tan^{-1}\frac{3}{4} \qquad \ldots(1)$$

Thus,

$$LHS=2\sin^{-1}\frac{3}{5}$$

$$=2\tan^{-1}\frac{3}{4} \qquad [from\ (1)]$$

$$=\tan^{-1}\left(\frac{2\times\frac{3}{4}}{1-\left(\frac{3}{4}\right)^2}\right)$$

$$=\tan^{-1}\left(\frac{24}{7}\right)$$

$$=RHS$$

Question 4:

Prove that $\sin^{-1}\frac{8}{17}+\sin^{-1}\frac{3}{5}=\tan^{-1}\frac{77}{36}$.

Solution:

Let $\sin^{-1}\frac{8}{17}=x\Rightarrow \sin x=\frac{8}{17}$

Then,

$$\cos x=\sqrt{1-\left(\frac{8}{17}\right)^2}=\sqrt{\frac{225}{289}}=\frac{15}{17}$$

Therefore,

$$\tan x=\frac{8}{15}$$

$$x=\tan^{-1}\frac{8}{15}$$

$$\sin^{-1}\frac{8}{17}=\tan^{-1}\frac{8}{15} \qquad \ldots(1)$$

Now, let $\sin^{-1}\frac{3}{5}=y\Rightarrow \sin y=\frac{3}{5}$
Then,

$$\cos y=\sqrt{1-\left(\frac{3}{5}\right)^2}=\sqrt{\frac{16}{25}}=\frac{4}{5}$$

Therefore,

$$\tan y=\frac{3}{4}$$
$$y=\tan^{-1}\frac{3}{4}$$
$$\sin^{-1}\frac{3}{5}=\tan^{-1}\frac{3}{4} \qquad \ldots(2)$$

Thus, by using (1) and (2)

$$\begin{aligned} LHS &= \sin^{-1}\frac{8}{17}+\sin^{-1}\frac{3}{5} \\ &= \tan^{-1}\frac{8}{15}+\tan^{-1}\frac{3}{4} \\ &= \tan^{-1}\left[\frac{\frac{8}{15}+\frac{3}{4}}{1-\frac{8}{15}\cdot\frac{3}{4}}\right] \\ &= \tan^{-1}\left[\frac{\frac{32+45}{60}}{\frac{60-24}{60}}\right] \\ &= \tan^{-1}\frac{77}{36} \\ &= RHS \end{aligned}$$

Question 5:

Prove that $\cos^{-1}\frac{4}{5}+\cos^{-1}\frac{12}{13}=\cos^{-1}\frac{33}{65}$.

Solution:

Let $\cos^{-1}\frac{4}{5}=x\Rightarrow \cos x=\frac{4}{5}$
Then,

$$\sin x = \sqrt{1 - \left(\frac{4}{5}\right)^2} = \frac{3}{5}$$

Therefore,

$$\tan x = \frac{3}{4}$$

$$x = \tan^{-1}\frac{3}{4}$$

$$\cos^{-1}\frac{4}{5} = \tan^{-1}\frac{3}{4} \qquad \ldots(1)$$

Now, let $\cos^{-1}\frac{12}{13} = y \Rightarrow \cos y = \frac{12}{13}$

Then,

$$\sin y = \frac{5}{13}$$

Therefore,

$$\tan y = \frac{5}{12}$$

$$y = \tan^{-1}\frac{5}{12}$$

$$\cos^{-1}\frac{12}{13} = \tan^{-1}\frac{5}{12} \qquad \ldots(2)$$

Thus, by using (1) and (2)

$$\begin{aligned}\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} &= \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{5}{12} \\ &= \tan^{-1}\left[\frac{\frac{3}{4} + \frac{5}{12}}{1 - \frac{3}{4}\cdot\frac{5}{12}}\right] \\ &= \tan^{-1}\left[\frac{56}{33}\right] \qquad \ldots(3)\end{aligned}$$

Now, let $\cos^{-1}\frac{33}{65} = z \Rightarrow \cos z = \frac{33}{65}$

Then,

$$\sin z = \sqrt{1 - \left(\frac{33}{65}\right)^2} = \frac{56}{65}$$

Therefore,

$$\tan z = \frac{33}{56}$$

$$z = \tan^{-1}\frac{56}{33}$$

$$\cos^{-1}\frac{33}{65} = \tan^{-1}\frac{56}{33} \qquad \ldots(4)$$

Thus, by using (3) and (4)

$$\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65}$$

Hence proved.

Question 6:

Prove that $\cos^{-1}\frac{12}{13} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{56}{65}$.

Solution:

Let $\cos^{-1}\frac{12}{13} = y \Rightarrow \cos y = \frac{12}{13}$

Then,

$$\sin y = \sqrt{1 - \left(\frac{12}{13}\right)^2} = \frac{5}{13}$$

Therefore,

$$\tan y = \frac{5}{12}$$

$$y = \tan^{-1}\frac{5}{12}$$

$$\cos^{-1}\frac{12}{13} = \tan^{-1}\frac{5}{12} \qquad \ldots(1)$$

Now, let $\sin^{-1}\frac{3}{5} = x \Rightarrow \sin x = \frac{3}{5}$

Then,

$$\cos x = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \frac{4}{5}$$

Therefore,

$$\tan x = \frac{3}{4}$$

$$x = \tan^{-1}\frac{3}{4}$$

$$\sin^{-1}\frac{3}{5} = \tan^{-1}\frac{3}{4} \qquad \ldots(2)$$

Now, let $\sin^{-1}\frac{56}{65} = z \Rightarrow \sin z = \frac{56}{65}$

Then,

$$\cos z = \sqrt{1-\left(\frac{56}{65}\right)^2} = \frac{33}{65}$$

Therefore,

$$\tan z = \frac{56}{33}$$

$$z = \tan^{-1}\frac{56}{33}$$

$$\sin^{-1}\frac{56}{65} = \tan^{-1}\frac{56}{33} \qquad \ldots(3)$$

Thus, by using (1) and (2)

$$\begin{aligned}
LHS &= \cos^{-1}\frac{12}{13} + \sin^{-1}\frac{3}{5} \\
&= \tan^{-1}\frac{5}{12} + \tan^{-1}\frac{3}{4} \\
&= \tan^{-1}\left[\frac{\frac{5}{12}+\frac{3}{4}}{1-\frac{5}{12}\cdot\frac{3}{4}}\right] \\
&= \tan^{-1}\left[\frac{\frac{20+36}{48}}{\frac{48-15}{48}}\right] \\
&= \tan^{-1}\left(\frac{56}{33}\right) \\
&= \sin^{-1}\frac{56}{65} \qquad \left[\text{Using}(3)\right] \\
&= RHS
\end{aligned}$$

Question 7:

Prove that $\tan^{-1}\frac{63}{16}=\sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5}$.

Solution:

Let $\sin^{-1}\frac{5}{13}=x \Rightarrow \sin x=\frac{5}{13}$

Then,

$$\cos x=\sqrt{1-\left(\frac{5}{13}\right)^2}=\frac{12}{13}$$

Therefore,

$$\tan x=\frac{5}{12}$$

$$x=\tan^{-1}\frac{5}{12}$$

$$\sin^{-1}\frac{5}{13}=\tan^{-1}\frac{5}{12} \qquad \ldots(1)$$

Now, let $\cos^{-1}\frac{3}{5}=y \Rightarrow \cos y=\frac{3}{5}$

Then,

$$\sin y=\sqrt{1-\left(\frac{3}{5}\right)^2}=\frac{4}{5}$$

Therefore,

$$\tan y=\frac{4}{3}$$

$$y=\tan^{-1}\frac{4}{3}$$

$$\cos^{-1}\frac{3}{5}=\tan^{-1}\frac{4}{3} \qquad \ldots(2)$$

Thus, by using (1) and (2)

$$
\begin{aligned}
RHS &= \sin^{-1}\frac{5}{12}+\cos^{-1}\frac{3}{5}\\
&= \tan^{-1}\frac{5}{12}+\tan^{-1}\frac{4}{3}\\
&= \tan^{-1}\left(\frac{\frac{5}{12}+\frac{4}{3}}{1-\frac{5}{12}\cdot\frac{4}{3}}\right)\\
&= \tan^{-1}\left(\frac{63}{16}\right)\\
&= LHS
\end{aligned}
$$

Question 8:

Prove that $\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$

Solution:

$$
\begin{aligned}
LHS &= \tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8}\\
&= \tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}\cdot\frac{1}{7}}\right)+\tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}\cdot\frac{1}{8}}\right)\\
&= \tan^{-1}\left(\frac{12}{34}\right)+\tan^{-1}\left(\frac{11}{23}\right)\\
&= \tan^{-1}\left(\frac{6}{17}\right)+\tan^{-1}\left(\frac{11}{23}\right)\\
&= \tan^{-1}\left(\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}\cdot\frac{11}{23}}\right)\\
&= \tan^{-1}\left(\frac{325}{325}\right)\\
&= \tan^{-1}(1)\\
&= \frac{\pi}{4}\\
&= RHS
\end{aligned}
$$

Question 9:

Prove that $\tan^{-1}\sqrt{x}=\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right), x\in[0,1]$.

Solution:

Let $x = \tan^2\theta$

Then,

$$\sqrt{x} = \tan\theta$$
$$\theta = \tan^{-1}\sqrt{x}$$

Therefore,

$$\left(\frac{1-x}{1+x}\right) = \frac{1-\tan^2\theta}{1+\tan^2\theta}$$
$$= \cos 2\theta$$

Thus,

$$RHS = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$$
$$= \frac{1}{2}\cos^{-1}(\cos 2\theta)$$
$$= \frac{1}{2}\times 2\theta$$
$$= \theta$$
$$= \tan^{-1}\sqrt{x}$$
$$= LHS$$

Question 10:

Prove that $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \frac{x}{2}, x \in \left(0, \frac{\pi}{4}\right)$.

Solution:

$$\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \frac{\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)^2}{\left(\sqrt{1+\sin x}\right)^2-\left(\sqrt{1-\sin x}\right)^2} \qquad (\textit{by rationalizing})$$
$$= \frac{(1+\sin x)+(1-\sin x)+2\sqrt{(1+\sin x)(1-\sin x)}}{1+\sin x-1+\sin x}$$
$$= \frac{2\left(1+\sqrt{1-\sin^2 x}\right)}{2\sin x} = \frac{1+\cos x}{\sin x}$$
$$= \frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}$$
$$= \cot\frac{x}{2}$$

Thus,

$$\begin{aligned} LHS &= \cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) \\ &= \cot^{-1}\left(\cot\frac{x}{2}\right) \\ &= \frac{x}{2} \\ &= RHS \end{aligned}$$

Question 11:

Prove that $\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)=\frac{\pi}{2}-\frac{1}{2}\cos^{-1}x, -\frac{1}{\sqrt{2}}\leq x\leq 1$

Solution:

Let $x=\cos 2\theta \Rightarrow \theta=\frac{1}{2}\cos^{-1}x$

Thus,

$$\begin{aligned} LHS &= \tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right) \\ &= \tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}\right) \\ &= \tan^{-1}\left(\frac{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}\right) \\ &= \tan^{-1}\left(\frac{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}\right) \\ &= \tan^{-1}\left(\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right) \\ &= \tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right) \\ &= \tan^{-1}1-\tan^{-1}(\tan\theta) \\ &= \frac{\pi}{4}-\theta \\ &= \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x \\ &= RHS \end{aligned}$$

Question 12:

Prove that $\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\frac{1}{3}=\frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}$

Solution:

$$LHS=\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\frac{1}{3}$$
$$=\frac{9}{4}\left(\frac{\pi}{2}-\sin^{-1}\frac{1}{3}\right)$$
$$=\frac{9}{4}\left(\cos^{-1}\frac{1}{3}\right) \qquad \ldots(1)$$

Now, let $\cos^{-1}\frac{1}{3}=x\Rightarrow\cos x=\frac{1}{3}$
Therefore,

$$\sin x=\sqrt{1-\left(\frac{1}{3}\right)^2}$$
$$=\frac{2\sqrt{2}}{3}$$
$$x=\sin^{-1}\frac{2\sqrt{2}}{3}$$
$$\cos^{-1}\frac{1}{3}=\sin^{-1}\frac{2\sqrt{2}}{3} \qquad \ldots(2)$$

Thus, by using (1) and (2)

$$\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\frac{1}{3}=\frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}$$

Hence proved.

Question 13:

Solve $2\tan^{-1}(\cos x)=\tan^{-1}(2\operatorname{cosec} x)$.

Solution:

It is given that $2\tan^{-1}(\cos x)=\tan^{-1}(2\operatorname{cosec} x)$

Since, $2\tan^{-1}(x)=\tan^{-1}\frac{2x}{1-x^2}$
Hence,

$$\Rightarrow \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right) = \tan^{-1}(2\operatorname{cosec} x)$$

$$\Rightarrow \left(\frac{2\cos x}{1-\cos^2 x}\right) = (2\operatorname{cosec} x)$$

$$\Rightarrow \frac{2\cos x}{\sin^2 x} = \frac{2}{\sin x}$$

$$\Rightarrow \cos x = \sin x$$

$$\Rightarrow \tan x = 1$$

$$\Rightarrow \tan x = \tan\frac{\pi}{4}$$

Therefore,

$x = n\pi + \frac{\pi}{4}$, where $n \in Z$.

Question 14:

Solve $\tan^{-1}\frac{1-x}{1+x} = \frac{1}{2}\tan^{-1} x, (x > 0)$

Solution:

Since $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\frac{x-y}{1+xy}$

Hence,

$$\Rightarrow \tan^{-1}\frac{1-x}{1+x} = \frac{1}{2}\tan^{-1} x$$

$$\Rightarrow \tan^{-1}1 - \tan^{-1} x = \frac{1}{2}\tan^{-1} x$$

$$\Rightarrow \frac{\pi}{4} = \frac{3}{2}\tan^{-1} x$$

$$\Rightarrow \tan^{-1} x = \frac{\pi}{6}$$

$$\Rightarrow x = \tan\frac{\pi}{6}$$

$$\Rightarrow x = \frac{1}{\sqrt{3}}$$

Question 15:

Solve $\sin(\tan^{-1} x), |x| < 1$ is equal to

(A) $\frac{x}{\sqrt{1-x^2}}$ (B) $\frac{1}{\sqrt{1-x^2}}$ (C) $\frac{1}{\sqrt{1+x^2}}$ (D) $\frac{x}{\sqrt{1+x^2}}$

Solution:

Let $\tan y = x$

Therefore,

$$\sin y = \frac{x}{\sqrt{1+x^2}}$$

Now, let $\tan^{-1} x = y$

Therefore,

$$y = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$$

Hence,

$$\tan^{-1} x = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$$

Thus,

$$\sin\left(\tan^{-1} x\right) = \sin\left(\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)\right)$$

$$= \frac{x}{\sqrt{1+x^2}}$$

Thus, the correct option is D.

Question 16:

Solve: $\sin^{-1}(1-x) - 2\sin^{-1} x = \frac{\pi}{2}$, then x is equal to

(A) $0, \frac{1}{2}$ (B) $1, \frac{1}{2}$ (C) 0 (D) $\frac{1}{2}$

Solution:

It is given that $\sin^{-1}(1-x) - 2\sin^{-1} x = \frac{\pi}{2}$

$$\Rightarrow \sin^{-1}(1-x) - 2\sin^{-1} x = \frac{\pi}{2}$$

$$\Rightarrow -2\sin^{-1} x = \frac{\pi}{2} - \sin^{-1}(1-x)$$

$$\Rightarrow -2\sin^{-1} x = \cos^{-1}(1-x) \qquad \ldots(1)$$

Let $\sin^{-1} x = y \Rightarrow \sin y = x$

Hence,

$$\cos y=\sqrt{1-x^2}$$
$$y=\cos^{-1}\left(\sqrt{1-x^2}\right)$$
$$\sin^{-1}x=\cos^{-1}\sqrt{1-x^2}$$

From equation (1), we have

$$-2\cos^{-1}\sqrt{1-x^2}=\cos^{-1}(1-x)$$

Put $x=\sin y$

$$\Rightarrow -2\cos^{-1}\sqrt{1-\sin^2 y}=\cos^{-1}(1-\sin y)$$
$$\Rightarrow -2\cos^{-1}(\cos y)=\cos^{-1}(1-\sin y)$$
$$\Rightarrow -2y=\cos^{-1}(1-\sin y)$$
$$\Rightarrow 1-\sin y=\cos(-2y)$$
$$\Rightarrow 1-\sin y=\cos 2y$$
$$\Rightarrow 1-\sin y=1-2\sin^2 y$$
$$\Rightarrow 2\sin^2 y-\sin y=0$$
$$\Rightarrow \sin y(2\sin y-1)=0$$
$$\Rightarrow \sin y=0,\frac{1}{2}$$

Therefore,

$$x=0,\frac{1}{2}$$

When $x=\frac{1}{2}$, it does not satisfy the equation.
Hence, $x=0$ is the only solution

Thus, the correct option is C.

Question 17:

Solve $\tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\frac{x-y}{x+y}$ is equal to

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{4}$ (D) $\frac{-3\pi}{4}$

Solution:

$$\tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\frac{x-y}{x+y}=\tan^{-1}\left[\frac{\frac{x}{y}-\frac{x-y}{x+y}}{1+\left(\frac{x}{y}\right)\left(\frac{x-y}{x+y}\right)}\right]$$

$$=\tan^{-1}\left[\frac{\frac{x(x+y)-y(x-y)}{y(x+y)}}{\frac{y(x+y)+x(x-y)}{y(x+y)}}\right]$$

$$=\tan^{-1}\left(\frac{x^2+xy-xy+y^2}{xy+y^2+x^2-xy}\right)$$

$$=\tan^{-1}(1)$$

$$=\tan^{-1}\left(\tan\frac{\pi}{4}\right)$$

$$=\frac{\pi}{4}$$

Thus, the correct option is C.